The Quest to Complete the History of the Universe – An Exploration of Cosmic Inflation and Cyclic Models

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"The substance of the universe is obedient and compliant; and the reason which governs it has in itself no cause for doing evil, for it has no malice, nor does it do evil to anything, nor is anything harmed by it. But all things are made and perfected according to this reason."

– Ch. 6, Meditations, Marcus Aurelius

To my grandparents

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It is times like these that make me truly understand the depth in the oft-quoted adage that one's accomplishments are not achieved in a vacuum. This year has involved nothing short of astronomical effort, and I certainly would not have been able to reach this far without the abundance of support behind me.

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Abstract

The field of cosmology is at an inflection point today. The standard hot Big Bang description is known to have problems that make it incongruous with the nearly homogeneous, isotropic and flat Universe that we observe. The paradigm of inflation was originally proposed to solve these problems, starting from any generic initial state. While it has been remarkably successful in resolving the horizon, flatness and large-scale structure formation problems, other issues have also come to light in recent years regarding its viability, especially regarding the level of fine-tuning that is expected to occur in order for inflation to begin and be able to explain the state of our Universe. Multiple alternative models have been proposed to explain the known problems of hot Big Bang cosmology as successfully as inflation, but with fewer drawbacks. In this dissertation, we review the mechanisms by which inflation solves the horizon and flatness problems, as well as that of large-scale structure formation through perturbations in the scalar sector. We also show how perturbations in the tensor sector lead to predictions of primordial gravitational waves. After presenting an example of embedding inflation within a fundamental string setup (i.e. boundary inflation), we discuss the issues faced by the paradigm. We then introduce bouncing cosmologies, with a focus on the cyclic/phoenix model of Turok, Steinhardt *et. al.* Specifically, we review the background dynamics and show how this model explains large-scale structure formation through the two-scalar entropic/isocurvature mechanism; we also contrast the prediction of a negligible spectrum of primordial gravitational waves with the corresponding prediction from inflation. We utilise problems faced by the older models as motivation for investigating nonsingular bounces, such as those found in the new cyclic model of Ijjas and Steinhardt. We consider a resolution of the background instability arising from the isocurvature mechanism, by way of a non-linear sigma model-type coupling between the two scalar fields, and examine the nonsingular bouncing phase. This necessarily involves a violation of the null energy condition and is realised using a modified gravity theory (which, in this specific case, is generalised Galileon/Horndeski theory). In particular, we pay attention to the expected constraints placed by perturbative unitarity during a period of NEC violation within a Horndeski model. We conclude by considering open questions in this model and avenues for future research.

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Introduction

1

1.1 The Question of Cosmic Evolution

Since time immemorial, man has looked to the heavens and thought most deeply about two questions – where did we come from and where are we going? While modern science has provided some rational direction to this line of thought, numerous civilizations from ancient times have come up with their own (albeit largely philosophical) ideas with respect to these questions. As an exemplar of the Ancient Greeks, Plato laid out in the *Timaeus* that the Universe was created by a supreme divine being, whose generation of order in the form of a static, eternal cosmos from pre-existing chaos was a direct product of intellect (thereby imposing mathematical order) and benevolence.

However, the notion of a cyclic Universe has been presented in ancient times as well. Hindu scriptures point to the holy trinity of Brahma (the Creator), Vishnu (the Preserver) and Shiva (the Destroyer) as architects of each epoch of the Universe's lifetime. Each deity's actions serve to create, preserve and destroy the Universe in turn, and the cycle repeats infinitely many times. In this sense, the question of the Universe having a singular beginning or end is rendered meaningless.

While the Greeks were among the first to attempt to understand the movement of heavenly bodies using mathematics, their ideas of the evolution of the Universe as a whole still called upon mythological elements. With Newton came the advent of the modern scientific age, and it was then that, for the first time, these questions could be seriously explored using mathematical rigour. Newton nevertheless also postulated a static, eternal Universe and this line of thought was continued by Einstein even in the framework of General Relativity, in the hope that one could avoid the question of a moment of Creation (along with its other myriad metaphysical implications). To ensure this, Einstein inserted into the field equations of General Relativity what he famously termed "his biggest blunder" – the cosmological constant. This so came to be known with the discoveries of Lemaître and Hubble of the recession of galaxies, which implied an expanding Universe. It was at that point that hot Big Bang cosmology, as we know it today, came to be born and with it, the issues that inflationary and alternative models seek to address.

To provide some context, let us consider the development of events from a historical perspective. The horizon, large-scale structure and flatness problems (as described in the next section) were known to be major roadblocks to a successful description of the Universe as far back as 1956 and 1969, with Rindler [1] and Dicke [2] making the first concrete formulations respectively. In order to solve these problems, the original idea of inflation was presented by Guth [3], where the primordial Universe was trapped in a false vacuum. Bubbles of true vacuum would appear and expand at the speed of light, and bubble nucleation would lead to the Universe decaying into a state of the true vacuum. However, amongst others, Guth himself realised that his model would not yield a transition to radiation domination and would not result in the required properties of homogeneity and isotropy [4]. The model of 'new inflation' was proposed by Linde [5], and independently by Albrecht and Steinhardt [6]. Here, the Universe would undergo approximately 60 e-folds of expansion as the inflaton scalar field rolled slowly down a potential hill. Then the decay of the scalar field from a false vacuum to the true minimum, i.e. from a state with non-zero energy density to one with zero energy density, would signal the end of the inflationary period and would lead to the (largely) homogeneous and flat universe we observe today. Furthermore, quantum-level perturbations in the scalar field would give rise to the density perturbations that eventually led to the formation of galaxies and other large-scale structures [7, 8]. The evidence of a near-scale invariant spectrum of density perturbations, provided by observational data [9], is seen as one of the crowning jewels of inflation's predictive power. This, together with the possibility that the inflaton could be identified with the Higgs particle [4] in an $SU(5)$ Grand Unified Theory picture [10], made inflation an extremely attractive proposal from both a theoretic and a cosmological point of view. However, with respect to this last point, the amplitude of the false vacuum would be much higher than a naïve GUT scale [11] and a non-minimal coupling of the Standard Model Higgs field to gravity (implying non-renormalisability and a forced interpretation as an effective field theory) would present possible issues concerning unitarity violation, although work is being done to explore this further (see for example [12]).

New inflation mandated that the driving potential would have to be exceptionally flat [13] and would only take place if certain initial conditions were satisfied [14]. To combat this, 'chaotic inflation' was proposed by Linde [15], wherein the initial conditions could be satisfied rather naturally and the fine-tuning problem would be resolved. However, the value of the inflaton field would have to be large, thereby preventing an effective field theory description

[16] and requiring a setup within a more fundamental picture, such as string theory. One would then have to consider the effect of non-renormalisable terms on the extent of inflation.

Since the question of the identity and origin of the inflaton scalar is still open, one may look to theories where scalar fields naturally arise and ask how inflation could be embedded within these theories. In particular, string theory provides instances where the origin of scalars can in fact be explained (rather than their presence being taken for granted). For example, we will consider cosmology in the setting of Hořava-Witten theory, which represents M-theory on the orbifold $S^1/\mathbb{Z}_2$ (as initially proposed by Waldram *et. al.* [17, 18, 19]). The 5-dimensional effective theory arises by reducing the full 11-dimensional theory on a Calabi-Yau 3-fold, and has the geometrical structure $\mathcal{M}_5 = S^1/\mathbb{Z}_2 \times \mathcal{M}_4$. The vacuum state is a BPS double 3-brane boundary wall solution, with each of the 3-branes identified as orbifold planes. Reducing from 5 to 4 dimensions yields an $\mathcal{N} = 1$ supergravity theory and the visible Universe can be thought of as living on one of these orbifold planes (the other plane contains the hidden fields in the model). In this case, a scalar field can arise as the volume modulus of the internal Calabi-Yau space and scalars can also be identified as partners of matter fields living on the orbifold boundary planes. This construction is of particular interest as the latter 'boundary scalars' ϕ_i can be used to drive inflation (the former 'bulk scalar' ϕ can also be used, but the setup we will consider does not explore this). A rigorous treatment of this subject, especially from an M-theory point of view, is beyond the scope of this dissertation, and we will only present the essential points required to visualise how inflation may arise from the fundamental theory.

Despite its seeming successes, the inflationary paradigm appears to have some key problems (which we will discuss in detail in Chapter 2), and the recognition of these was amidst the driving factors that led to constructions of alternative models [20]. Prime among these was the ekpyrotic/cyclic model of Turok, Steinhardt *et. al.* which, interestingly, utilised the same underlying theoretic construction as the boundary inflation proposal in [19], i.e. a BPS double 3-brane vacuum state in Hŏrava-Witten theory with each of the 3-branes corresponding to an orbifold plane on which we have the visible Universe and hidden sector respectively. In the original ekpyrotic scenario [21], a bulk brane peeled away from the hidden brane spontaneously and moved slowly towards the visible brane. The collision of the bulk and visible branes was identified with the creation of the Universe in the hot Big Bang phase at a finite temperature. An evolution of this model was the cyclic Universe [22, 23], in which the visible and hidden branes move towards each other along the (orbifold) extra dimension. The brane collision is associated with the Big Bang, after which they move apart again until the potential turns slightly positive, causing a transition to contraction. This cycle repeats eternally, from which the model derives its name. Note that one distinction from the boundary inflation scenario that we will consider is that the scalar field driving the cosmic evolution is the bulk scalar (rather than the boundary scalar/s). Furthermore, in contrast to inflation, where large-scale structure formation is explained by the growth of perturbations in the inflaton scalar field (after the Big Bang), the density perturbations are generated in the cyclic Universe during the contracting phase. To understand the problem of generating these perturbations and matching to perturbations in the expanding phase, see for example [24, 25, 26, 27] and references therein. The role of dark energy is of especial significance in this cyclic model – in contrast to the standard Λ-CDM model cosmology, where dark energy is an *ad hoc* insertion, it is in fact the bulk scalar ϕ that serves as the source of the accelerated expansion in the present epoch. The transition to contraction occurs when ϕ rolls in the negative direction, passes the point of zero total energy and continues towards $-\infty$ (which signifies the Big Crunch-Big Bang transition).

Note that the model described is not the only version of a bouncing cosmology – the matter bounce scenario uses the duality of the cosmological fluctuations in a contracting and an expanding phase to posit a bounce of the Universe, whereas the pre-Big Bang model uses superstring theory to connect contracting and expanding solutions for the scale factor. For completeness, we note that a model of string gas cosmology has also been proposed as an alternative to inflation [28]; however, we will not consider this here as it does not solve the flatness problem by itself. We will discuss the matter bounce and pre-Big Bang models briefly in Chapter 3; see [29] for an introductory review.

These developments bring us to the mid 2010s. Up to this point, the successful alternatives to inflation involved borrowing ingredients from M-theory or extra dimensions etc., which come with their own set of foundational issues[1]. A radically different concept of a cyclic Universe was presented in [31], where the Hubble parameter oscillates between large negative and large positive (but finite and sub-Planckian) values, while the scale factor $a(t)$ undergoes a net de Sitter phase globally. This is achieved by combining an ekpyrotic phase with General Relativity, with the bounce phase realised in the context of a cubic Galileon/Horndeski theory [32, 33, 34]. In essence, this implies that the evolution of the Universe could be classical at all points in this model and the bounce would shield it from a singularity. We will undertake a detailed exploration of this model in Chapter 4, and contrast it with the successes and failures of inflation, as well as those of its cyclic predecessors. Given the classical nature of this model, it will be interesting to see how quantum requirements

[1] An anamorphic cosmological model was also put forth in 2015 [30], but we will not consider this here.

affect it or the underlying theory. Specifically, we will consider whether unitarity is brought into question at tree level, and if any assumptions (beyond those already taken) are required to enforce it.

1.2 Outline

The aim of this dissertation is to present advances in early Universe cosmology in light of modern theoretical physics. We introduce the questions presented by the hot Big Bang Model of cosmology and proceed to answer them using two distinct paradigms – cosmic inflation and cyclic theory. We will also discuss open questions relating to each of these two models. This work is aimed at a reader with a background in general relativity and quantum field theory, and some familiarity with string theory. For detailed derivations of advanced concepts relating to the latter, we will direct the reader to the appropriate references.

The outline of this dissertation is as follows – in Chapter 2, we start from the state of the hot Big Bang model and examine its implications for the early Universe. Specifically, we deduce the need for a mechanism to explain why the Universe appears the way it does in the present day. We then present cosmic inflation and the means by which it proposes answers to the aforementioned questions. The dynamics of the single-field slow-roll mechanism are presented. The power spectrum and spectral index for scalar and tensor perturbations are calculated in the basic set-up, and essential roadblocks are discussed. Given these roadblocks, brane inflation is presented as a potential way to embed inflation in a UV-complete theory of gravity. We then discuss the major problems facing inflation and the reasons for considering alternative models of early Universe cosmology.

In Chapter 3, we consider cyclic models of cosmic evolution, with particular emphasis on the construction of Turok, Steinhardt *et. al.* It will be shown that the underlying theoretical set-up for the brane inflation scenario presented in Chapter 2 is essentially the same as that used for this cyclic mechanism. As in Chapter 2, the power spectrum and spectral index for scalar and tensor perturbations are calculated, and this is presented as a means of observationally distinguishing the two models. Finally, we will discuss the issues plaguing these 'older' cyclic models, which leaves room for considering advances in this direction.

Chapter 4 presents the newest iteration of a cyclic mechanism (postulated by Ijjas and Steinhardt) and one of the most intriguing, as it is defined entirely in terms of a classical framework (as opposed to a string/M-theory construction for the models discussed in Chapter 3). We will show how this 'new' cyclic model addresses each of the problems with the older models. In particular, this model will be seen to have particular appeal *vis a vis* inflation as it provides a unified view of cosmic evolution (by describing a link between the early Universe and the observed accelerated expansion in the present epoch, a feature it shares with the model of Turok and Steinhardt). We explore the question of unitarity in the regime of the non-singular bounce, which we present as a proof of concept of new work relating to this construction (and hope to investigate more fully in the future). We close the chapter by discussing various open questions that remain.

We conclude with Chapter 5, where we reiterate the key results of this dissertation, and show possible avenues that we hope to explore in the future.

Throughout this dissertation, we will work in reduced Planck units, with $\hbar = c = 1$ and $1/M_{pl}^2 = 8\pi G = 1$ [2]. The metric signature we choose is the mostly-plus convention $(-+++)$.

[2]Except in specific sections, where we re-introduce M_{pl} to make certain quantities manifestly dimensionless.

The Inflationary Paradigm

2

2.1 The Problems of Hot Big Bang Cosmology

It is well-established from observational data that the large scale Universe today can be described by the Friedmann-Lemaître-Robertson-Walker (FLRW) metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2 \left(\frac{\mathrm{d}r^2}{1 - kr^2} + r^2 \, \mathrm{d}\Omega^2 \right), \tag{2.1.1}$$

where $a(t)$ is the scale factor, k is the spatial curvature and $\mathrm{d}\Omega^2 = r^2(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2)$. Present data points to the Universe being spatially flat, i.e. $k = 0$, and we will hence restrict our focus to this case in the rest of this dissertation. The evolution of the scale factor is described by the first and second Friedmann equations (the latter is also known as the Raychaudhuri equation) [35, 36]

$$H^2 \equiv \left(\frac{\dot{a}}{a} \right)^2 = \frac{\rho}{3}, \tag{2.1.2}$$

$$\frac{\ddot{a}}{a} = -\frac{1}{2} \left(\frac{\rho}{3} + P \right), \tag{2.1.3}$$

where $H \equiv \frac{\dot{a}}{a}$ is the Hubble parameter, ρ is the total contribution of pressure-free matter (dust) and P is the total contribution of radiation to the stress-energy tensor $T_{\mu\nu}$ [1]. These two equations can be straightforwardly derived by substituting the FLRW metric and the stress-energy tensor for a perfect fluid into the tt and ij components of Einstein's field equations of gravitation. Moreover, one can combine them to yield the conservation equation

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0. \tag{2.1.4}$$

[1]Dots refer to derivatives with respect to coordinate time, $\dot{a} = \partial_t a$.

Tracing the evolution of the scale factor backwards in time, we can infer that there must have a point in the history of the Universe at which it was infinitesimally small ($a \to 0$) and infinitely hot (since $T \sim 1/a$). This origin from a singularity gives rise to the hot Big Bang model. However, reconciling the presently-observed form of the Universe with a singular origin leads us to some well-known problems:

Horizon Problem

The observation of the Cosmic Microwave Background (CMB) at a temperature of 2.728 K (with a root mean square (rms) deviation of 1.8×10^{-5} K [37]) implies that greatly separated regions of space appear to have the same physical properties. However, we consider the following: the greatest comoving distance from which a signal (travelling at the speed of light) can be received by an observer is given by the comoving particle horizon [38]

$$d_{ph}(\tau) = \tau - \tau_i = \int_{t_i}^{t} \frac{\mathrm{d}t}{a(t)}, \tag{2.1.5}$$

where τ refers to conformal time and the initial Big Bang singularity is at $\tau_i = t_i = 0$. One can deduce that the conformal time elapsed between the singularity and the time when the CMB radiation formed (known as the time of recombination or last scattering) is much smaller than the conformal age of the Universe ($\tau_{rec} \ll \tau_0$), and so we would expect there to be causally disconnected patches in the sky, since a signal travelling at the speed of light from one region could not possibly reach the other. The full extent of this problem is presented when we realise that the CMB comprises more than 4×10^5 causally disconnected regions [38]. This poses the question of how the temperature came to be so uniform when there was no possible way these regions could have interacted. This is known as the horizon or homogeneity problem.

Flatness Problem

As pointed out at the beginning of the section, the present Universe is nearly flat, and $k = 0$ can be used in (2.1.1) at a high confidence level. Considering the critical density $\rho_c = \frac{3H^2}{8\pi}$, this means that the average density of the Universe must be very close to ρ_c (as any small deviation in ρ would cause a deviation in k, which in turn would grow as the Universe expands). In fact, WMAP (Wilkinson Microwave Anisotropy Probe) data [9] constrains the ratio $\Omega = \rho/\rho_c$ to be 1 today, with the tightest bound on the deviation to be $|\Omega - 1| = 0.011$.

One can relate Ω today to that at some earlier time using (2.1.2), so that

$$(\Omega(t) - 1) = (\Omega_0 - 1)\frac{(a_0 H_0)^2}{(aH)^2}, \tag{2.1.6}$$

where the subscript 0 refers to the values of the various parameters today. Then by considering that $a(t) \propto t^{1/2}$ during radiation domination and $a(t) \propto t^{2/3}$ during matter domination, we can deduce that the deviation $|\Omega - 1|$ would have had to have been at the order of 10^{-60} just after the Big Bang. This implies an incredible level of fine-tuning, and is known as the flatness problem.

Formation of Large-Scale Structure

This is related to the horizon problem discussed above. Specifically, galaxies and galaxy clusters are found to have correlations which are nontrivial [39], on scales comparable to the time of matter-radiation equality [40] (recalling that $c = 1$). Assuming that the fluctuations in density were generated much before this time, these correlations cannot be reconciled with causality [41]. Hence, modern (standard) cosmology presents the problem of what generated the initial density perturbations, and what causes these non-random correlations.

Origin of the Universe

The initial singularity representing the origin of the known Universe is one of the major outstanding questions in cosmology today. A mechanism that can explain generation of the cosmos is hitherto unknown, although attempts have been made at finding solutions or circumventing the problem altogether. For example, Vilenkin proposed a model [42, 43] in which the Universe could be created by quantum tunneling from 'nothing', thereby solving the singularity issue. Alternatively, as introduced in Chapter 1, Steinhardt and Ijjas have postulated a non-singular bounce within a cyclic model [31] where the Hubble parameter (and hence the energy scale, which goes as $\sim H^2$) transitions from large and negative to large and positive (i.e., that representing a contracting Universe to an expanding Universe) at sub-Planckian values. We will explore this idea, as an alternative to the inflationary paradigm, in greater detail in Chapter 4.

2.2 The Inflationary Paradigm

Inflation relies on the Universe undergoing a period of accelerated expansion between the initial singularity and the following radiation-dominated epoch. This period can be approximated as a de Sitter phase, and is realised by the driving fluid having an equation of state $w \simeq -1$. We can visualise inflation as representing a period of reducing Hubble radius

$$\frac{\mathrm{d}}{\mathrm{d}t}\,(aH)^{-1} < 0, \tag{2.2.1}$$

so that the Hubble radius would have been greater than the particle horizon (2.1.5). It is also immediately evident that (2.2.1) represents accelerated expansion of the scale factor, $\ddot{a} > 0$. In fact, using this interpretation, one can show that the singularity gets pushed to $\tau \to -\infty$ in terms of conformal time. This means that $|\tau_e - \tau_i| \gg \tau_0 - \tau_{rec}$, where τ_i, τ_e, τ_{rec} and τ_0 mark the conformal times corresponding to the start of inflation, end of inflation, time of recombination and present time respectively. As a result, the light cones of greatly separated patches of space intersect in the past and there is no issue with causality, thus solving the horizon problem. If we assume that the Universe had some nonzero spatial curvature k to start with, the Friedmann equation implies that this would have been diluted away by a factor of a^{-2} during the inflationary period, thereby resolving the flatness problem as well.

The large-scale structure formation problem is resolved through the following – the density fluctuations in the CMB were formed through quantum perturbations in the scalar field driving inflation, which were frozen out as super-horizon correlations as the Hubble radius shrank. As inflation ended and the Hubble radius expanded, they re-entered the horizon as classical fluctuations. The modes grew and formed the large-scale structures such as galaxies that we observe today.

By relating the reheating temperature (the maximum temperature at the start of the hot Big Bang phase) to that today and using the condition that the Hubble radius was shrinking during inflation, we can show that the minimum number of e-folds required for inflation to solve the problems of hot Big Bang cosmology is [38]

$$N = \ln\left(\frac{a_e}{a_i}\right) > 64 + \ln\left(\frac{T_R}{10^{15}\,\mathrm{GeV}}\right), \tag{2.2.2}$$

where T_R is the reheating temperature and 10^{15} GeV is the reference value introduced for T_R. We infer from this that the Universe must have undergone approximately 60 e-folds of expansion during inflation.

2.3 Scalar Field Dynamics

Recall that the action for a scalar field in a potential (minimally coupled to a general curved spacetime) [38] is

$$S_\phi = \int \mathrm{d}^4x \sqrt{-g} \left[-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi - V(\phi) \right] \tag{2.3.1}$$

Then varying with respect to the metric yields the energy-momentum tensor for the scalar field

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_\phi}{\delta g^{\mu\nu}} = \partial_\mu \phi \, \partial_\nu \phi - g_{\mu\nu} \left(\frac{1}{2} g^{\rho\sigma} \partial_\rho \phi \, \partial_\sigma \phi + V(\phi) \right) \tag{2.3.2}$$

Now, taking that $g_{\mu\nu}$ is the FLRW metric (2.1.1) and the scalar field is homogeneous $\phi(t, \mathbf{x}) = \phi(t)$, we have that ϕ can be approximated as a perfect fluid and $T_{\mu\nu} = \mathrm{diag}(\rho, P, P, P)$. Then the energy density and pressure induced are given by the tt and ij components of the energy-momentum tensor respectively

$$\begin{aligned} \rho_\phi = T_{tt} &= \frac{1}{2}\dot{\phi}^2 + V(\phi), \\ T_{ij} = P_\phi \gamma_{ij} \;\rightarrow\; P_\phi &= \frac{1}{2}\dot{\phi}^2 - V(\phi), \end{aligned} \tag{2.3.3}$$

where γ_{ij} is the spatial part of the metric. Then the equation of state is

$$w = \frac{P_\phi}{\rho_\phi} = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)} = -1 \text{ if } V \gg \dot{\phi}^2, \tag{2.3.4}$$

as required for an accelerating expansion of the Universe. Furthermore, varying the action (2.3.1) with respect to the scalar ϕ yields its equation of motion

$$\frac{\delta S_\phi}{\delta \phi} = 0 \;\rightarrow\; \ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial \phi} = 0 \tag{2.3.5}$$

This can also be obtained as the $\nu = t$ component of the (covariant) conservation of the energy-momentum tensor, $\nabla_\mu T^{\mu\nu} = 0$.

Let us define the parameter

$$\varepsilon \equiv -\frac{\dot{H}}{H^2} = \frac{\frac{1}{2}\dot{\phi}^2}{M_{pl}^2 H^2} = -\frac{\mathrm{d}\ln H}{\mathrm{d}N} \tag{2.3.6}$$

From the fact that the Hubble radius is reducing during inflation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left((aH)^{-1}\right) = -\frac{1}{(aH)^2}\left(\dot{a}H + a\dot{H}\right) = -\frac{1}{a}(1-\varepsilon) < 0 \;\rightarrow\; \varepsilon < 1 \tag{2.3.7}$$

Then inflation will take place if the kinetic term $\frac{1}{2}\dot{\phi}^2$ only has a small contribution to the total energy $\rho_\phi = \frac{3H^2}{8\pi} = 3M_{pl}^2 H^2$. This is referred to as slow-roll inflation. In order for this condition to remain true during the entire period of inflation, the acceleration of the scalar field should remain small. Let us define a second parameter to quantify this,

$$\delta \equiv -\frac{\ddot{\phi}}{H\dot{\phi}} \tag{2.3.8}$$

By taking the time derivative of ε

$$\dot{\varepsilon} = \frac{\dot{\phi}\ddot{\phi}}{M_{pl}^2 H^2} - \frac{\dot{\phi}^2 \dot{H}}{M_{pl}^2 H^3},$$

and inserting into $\eta \equiv \frac{\mathrm{d}\ln\varepsilon}{\mathrm{d}N} = \frac{\dot{\varepsilon}}{H\varepsilon}$, we obtain

$$\eta = \frac{\dot{\varepsilon}}{H\varepsilon} = 2\frac{\ddot{\phi}}{H\dot{\phi}} - 2\frac{\dot{H}}{H^2} = 2(\varepsilon - \delta) \tag{2.3.9}$$

Then the slow-roll approximation amounts to $\{\varepsilon, |\delta|\} \ll 1 \;\rightarrow\; \{\varepsilon, |\eta|\} \ll 1$.

2.4 Spectrum of Scalar Perturbations

As mentioned previously, inflation solves the problem of large-scale structure formation by quantum perturbations in the inflaton scalar field generating a scale-invariant power spectrum. Working to linear order, we will now see how this comes about. We perturb the metric and scalar field about their homogeneous background solutions

$$\phi(t, \mathbf{x}) = \bar{\phi}(t) + \delta\phi(t, \mathbf{x}), \quad g_{\mu\nu}(t, \mathbf{x}) = \bar{g}_{\mu\nu}(t) + \delta g_{\mu\nu}(t, \mathbf{x}) \tag{2.4.1}$$

Under a scalar-vector-tensor (SVT) decomposition, the metric becomes

$$\mathrm{d}s^2 = -(1 + 2\Phi)\,\mathrm{d}t^2 + 2aB_i\,\mathrm{d}x^i\,\mathrm{d}t + a^2\left[(1 - 2\Psi)\delta_{ij} + E_{ij}\right]\mathrm{d}x^i\,\mathrm{d}x^j, \tag{2.4.2}$$

where

$$B_i \equiv \partial_i B - S_i, \quad \partial_i S^i = 0$$

and

$$E_{ij} \equiv 2\partial_{ij}E + 2\partial_{(i}F_{j)} + h_{ij}, \quad \partial_i F^i = 0,\, h^i_i = \partial^i h_{ij} = 0$$

The vector perturbations can be neglected since they will decay away during the inflationary period. If we take the gauge transformations

$$t \to t + \alpha, \quad x^i \to \delta^{ij}\partial_j \beta, \tag{2.4.3}$$

then the scalar components of the metric perturbations transform as

$$\begin{gathered} \Phi \to \Phi - \dot{\alpha}, \quad \Psi \to \Psi + H\alpha \\ B \to B + a^{-1}\alpha - a\dot{\beta}, \quad E \to E - \beta \end{gathered} \tag{2.4.4}$$

Meanwhile, the tensor components are gauge-invariant. On the right-hand side of the Einstein equations, the components of the energy-momentum tensor become

$$\begin{gathered} T^0_0 = -(\bar{\rho} + \delta\rho), \quad T^0_i = (\bar{\rho} + \bar{P})\, a\, v_i \\ T^i_0 = -\frac{1}{a}(\bar{\rho} + \bar{P})\left(v^i - B^i\right), \quad T^i_j = \delta^i_j(\bar{P} + \delta P) + \Sigma^i_j \end{gathered} \tag{2.4.5}$$

Taking the momentum density as $\partial_i(\delta q) \equiv (\bar{\rho} + \bar{P})v_i$, the transformations are –

$$\delta\rho \to \delta\rho - \dot{\bar{\rho}}\alpha, \quad \delta P \to \delta P - \dot{\bar{P}}\alpha, \quad \delta q \to \delta q + (\bar{\rho} + \bar{P})\alpha \tag{2.4.6}$$

As with the tensor components for the SVT decomposition of the metric, the anisotropic stress Σ^i_j is gauge-invariant. Finding gauge-invariant combinations of the metric and energy-momentum (i.e. matter) perturbations [44] ensures that the gauge choice does not affect the conclusions. One of these is the curvature perturbation on hypersurfaces of uniform density [45]

$$-\zeta \equiv \Psi + \frac{H}{\dot{\bar{\rho}}}\delta\rho \tag{2.4.7}$$

A geometrical interpretation of this quantity is that it is a measure of the spatial curvature of hypersurfaces with constant density, $R^{(3)} = 4\nabla^2\Psi/a^2$. Using the required transfer functions, we can relate (2.4.7) to an expression in terms of the perturbation of the inflaton scalar field

$$-\zeta \approx \Psi + \frac{H}{\dot{\bar{\phi}}}\delta\phi \tag{2.4.8}$$

We also have the comoving curvature perturbation

$$\mathcal{R} \equiv \Psi - \frac{H}{\bar{\rho} + \bar{P}}\delta q \tag{2.4.9}$$

Using $T^0_i = -\dot{\bar{\phi}}\partial_i \delta\phi$ during inflation [46], this becomes

$$\mathcal{R} = \Psi + \frac{H}{\dot{\bar{\phi}}}\delta\phi \tag{2.4.10}$$

Geometrically, $\mathcal{R}$ is a measure of the spatial curvature of comoving hypersurfaces (i.e. with constant ϕ), and the perturbation in $\mathcal{R}$ is related to the spatially-varying time delay for the end of inflation $\delta t(\mathbf{x})$, which is caused by $\delta\phi$. Using the linearised Einstein equations (see Appendix A of [46]), we can find a relation between these two gauge-invariant quantities

$$-\zeta = \mathcal{R} + \frac{k^2}{(aH)^2}\frac{2\bar{\rho}}{3(\bar{\rho}+\bar{P})}\Psi_B, \tag{2.4.11}$$

where

$$\Psi_B \equiv \Psi + a^2 H\left(\dot{E} - \frac{B}{a}\right) \tag{2.4.12}$$

is one of the Bardeen potentials. Both ζ and $\mathcal{R}$ are conserved in the late-time limit. In fact they are equal in this limit, as well as during slow-roll inflation.

For completeness, we mention that there is a third gauge-invariant quantity, which is the inflaton perturbation on spatially flat slices

$$Q \equiv \delta\phi + \frac{\dot{\bar{\phi}}}{H}\Psi \tag{2.4.13}$$

In the following calculation, we will find the power spectrum of the comoving curvature perturbations, which (as mentioned above) is related to the spectrum of scalar perturbations. Our starting point is the action for gravity with a minimally-coupled scalar field

$$S = S_{EH} + S_\phi = \int \mathrm{d}^4x\,\sqrt{-g}\left(\frac{1}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - V(\phi)\right), \tag{2.4.14}$$

where $g = \det(g_{\mu\nu})$ and R is the Ricci scalar.

We fix temporal and spatial reparameterisations using the (comoving) gauge choices [47]

$$\delta\phi = 0,\;\; g_{ij} = a^2\left[(1-2\mathcal{R})\delta_{ij} + h_{ij}\right],\;\; \partial_i h_{ij} = h^i_i = 0, \tag{2.4.15}$$

where ϕ is unperturbed and all scalar degrees of freedom are parameterised by the metric fluctuation $\mathcal{R}(t,\mathbf{x})$. One significant point to note is that $\mathcal{R}$ remains constant outside the horizon, so the region of interest can be restricted to horizon crossing. Expanding the action (2.4.14) to second order in $\mathcal{R}$ (Appendix B of [46]), we obtain

$$S_{(2)} = \frac{1}{2}\int \mathrm{d}^4x\, a^3 \frac{\dot{\phi}^2}{H^2}\left[\dot{\mathcal{R}}^2 - a^{-2}\left(\partial_i \mathcal{R}\right)^2\right] \tag{2.4.16}$$

We now define the Mukhanov variable v by

$$v \equiv z\mathcal{R}, \quad z^2 \equiv a^2\frac{\dot{\phi}^2}{H^2} = 2a^2\varepsilon \tag{2.4.17}$$

Then, by explicitly bringing out the time-dependence and moving from coordinate time t to conformal time τ, the second-order action $S_{(2)}$ becomes (after integration by parts)

$$S_{(2)} = \frac{1}{2}\int \mathrm{d}\tau\, \mathrm{d}^3x \left[(v')^2 - (\partial_i v)^2 + \frac{z''}{z}v^2\right], \tag{2.4.18}$$

where a prime refers to a derivative with respect to conformal time, $v' = \partial_\tau v$. By defining the Fourier transform of the Mukhanov variable as

$$v(\tau, \mathbf{x}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\, v_{\mathbf{k}}(\tau)\, e^{i\mathbf{k}\cdot\mathbf{x}} \tag{2.4.19}$$

and varying the action with respect to $v_{\mathbf{k}}$, we obtain the Mukhanov-Sasaki equation

$$v_k'' + \left(k^2 - \frac{z''}{z}\right) v_k = 0 \tag{2.4.20}$$

We proceed to canonically quantise the theory by promoting the field v and its conjugate momentum to operators. Specifically,

$$v \to \hat{v} = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\left[v_k(\tau)\hat{a}_k\, e^{i\mathbf{k}\cdot\mathbf{x}} + v_k^*(\tau)\hat{a}_k^\dagger\, e^{-i\mathbf{k}\cdot\mathbf{x}}\right] \tag{2.4.21}$$

Equivalently, we could take

$$v_k \to \hat{v}_k = v_k(\tau)\,\hat{a}_{\mathbf{k}} + v_{-k}^*(\tau)\,\hat{a}_{-\mathbf{k}}^\dagger$$

The ladder operators obey the usual commutation relations

$$\left[\hat{a}_{\mathbf{k}}, \hat{a}_{\tilde{\mathbf{k}}}^\dagger\right] = (2\pi)^3\,\delta^{(3)}(\mathbf{k} - \tilde{\mathbf{k}}) \tag{2.4.22}$$

iff the mode functions are normalised as

$$\langle v_k, v_k \rangle \equiv i \left(v_k^* v_k' - v_k^{*\prime} v_k \right) = 1 \tag{2.4.23}$$

We must now choose a vacuum state for these modes. Following [46], we take the standard choice of the Minkowski vacuum for a comoving observer in the far past, with $\tau \to -\infty$, $|k\tau| \gg 1$, $k \gg aH$. In this limit, the Mukhanov-Sasaki equation reduces to

$$v_k'' + k^2 v_k = 0$$

A unique solution then exists if the vacuum is the minimum energy state, so we impose the following initial condition

$$\lim_{\tau \to -\infty} v_k = \frac{e^{-ik\tau}}{\sqrt{2k}} \tag{2.4.24}$$

By solving the Mukhanov-Sasaki equation using this initial condition, we obtain the Bunch-Davies mode function

$$v_k(\tau) = \frac{1}{\sqrt{2k}} \left(1 - \frac{i}{k\tau} \right) e^{-ik\tau} \tag{2.4.25}$$

Then the corresponding state is the vacuum state of the theory, and is known as the Bunch-Davies vacuum. The initial condition (2.4.24) and the normalisation condition (2.4.23) are then the two boundary conditions which fix the mode functions for all scales.

We will now calculate the power spectrum of curvature perturbations to first order in the slow-roll approximation. By differentiating z with respect to conformal time, we can show that

$$\frac{z'}{z} = \mathcal{H} \left(1 + \frac{\eta}{2} \right) \tag{2.4.26}$$

exactly and

$$\frac{z''}{z} \approx \mathcal{H}^2 \left(2 - \varepsilon + \frac{3}{2}\eta \right) \tag{2.4.27}$$

to first order in the slow-roll approximation, where $\mathcal{H} = a'/a$ is the conformal Hubble parameter. Then, since ε is approximately constant during inflation, we can integrate $\varepsilon = 1 - \frac{\mathcal{H}'}{\mathcal{H}^2}$ to obtain

$$\mathcal{H} = -\frac{1}{\tau}(1 + \varepsilon) \tag{2.4.28}$$

Using this expression in (2.4.27) yields

$$\frac{z''}{z} = \frac{1}{\tau^2} \left[2 + 3 \left(\varepsilon + \frac{\eta}{2} \right) \right] \tag{2.4.29}$$

Then the Mukhanov-Sasaki equation (2.4.20) becomes

$$v_k'' + \left(k^2 - \frac{\nu^2 - 1/4}{\tau^2}\right) v_k = 0, \tag{2.4.30}$$

where $\nu \equiv \frac{3}{2} + \varepsilon + \frac{\eta}{2}$. Using the boundary conditions (2.4.23) and (2.4.24), this has the the solution

$$v_k(\tau) = \frac{\sqrt{\pi}}{2} \sqrt{-\tau}\, H_\nu^{(1)}(-k\tau), \tag{2.4.31}$$

where $H_\nu^{(1)}$ is a Hankel function of the first kind, which behaves as

$$\lim_{k\tau \to -\infty} |H_\nu^{(1)}(-k\tau)| = \sqrt{\frac{2}{\pi}}\, \frac{1}{-k\tau}\, e^{-ik\tau} \tag{2.4.32}$$

at early times and

$$\lim_{k\tau \to 0} |H_\nu^{(1)}(-k\tau)|^2 = \frac{2^{2\nu}\Gamma(\nu)^2}{\pi^2} (-k\tau)^{-2\nu} \approx \frac{2}{\pi}(-k\tau)^{-2\nu} \tag{2.4.33}$$

at late times. Before finding the power spectrum of $\mathcal{R} = -v_k/z$, we require $z = z(\tau)$. We substitute (2.4.28) into (2.4.26) and integrate to obtain

$$z(\tau) = z_* \left(\frac{\tau}{\tau_*}\right)^{1/2-\nu}, \tag{2.4.34}$$

where τ_* is some reference conformal time. For convenience, we will take $\tau_* = -k_*^{-1}$, which is the conformal time corresponding to the moment of horizon crossing for a mode with wavenumber k_*. We are now able to find the power spectrum of $\mathcal{R}$ as

$$\begin{aligned} \Delta_{\mathcal{R}}^2(k) &= \frac{k^3}{2\pi^2} \frac{1}{z^2(\tau)} |v_k(\tau)|^2 \\ &= \frac{k^3}{2\pi^2} \frac{1}{2\varepsilon_* a_*^2} (-k_*\tau)^{2\nu-1} \frac{\pi}{4}(-\tau)|H_\nu^{(1)}(-k\tau)|^2 \end{aligned} \tag{2.4.35}$$

By taking $k_* = a_* H_*$, using the late-time limit of the Hankel function (2.4.33) and explicitly replacing $1/M_{pl}^2$, we obtain

$$\Delta_{\mathcal{R}}^2(k) = \frac{1}{8\pi^2\varepsilon_*} \frac{H_*^2}{M_{pl}^2} \left(\frac{k}{k_*}\right)^{3-2\nu} = A_s \left(\frac{k}{k_*}\right)^{3-2\nu}, \tag{2.4.36}$$

where A_s is the amplitude of the spectrum. The scalar spectral index is defined as

$$n_s - 1 \equiv \frac{\mathrm{d}\ln \Delta^2_{\mathcal{R}}(k)}{\mathrm{d}\ln k} \tag{2.4.37}$$

Using $k = aH$, this becomes

$$\begin{aligned} n_s - 1 = \frac{\mathrm{d}\ln \Delta^2_{\mathcal{R}}}{\mathrm{d}\ln(aH)} &\approx \frac{\mathrm{d}\ln \Delta^2_{\mathcal{R}}}{\mathrm{d}\ln a} \\ &= \frac{\mathrm{d}\ln \Delta^2_{\mathcal{R}}}{H\,\mathrm{d}t} = \frac{2\,\mathrm{d}\ln H}{H\,\mathrm{d}t} - \frac{\mathrm{d}\ln \varepsilon}{H\,\mathrm{d}t} \\ &= 2\frac{\dot{H}}{H^2} - \frac{\dot{\varepsilon}}{H\varepsilon} = -2\varepsilon - \eta \end{aligned} \tag{2.4.38}$$

Then using the definition of ν above, we have that the exponent in (2.4.36) is

$$3 - 2\nu = -2\varepsilon - \eta = n_s - 1$$

Hence, we find that the power spectrum of the curvature perturbations follows a power law

$$\Delta^2_{\mathcal{R}}(k) = A_s \left(\frac{k}{k_*}\right)^{n_s - 1} \tag{2.4.39}$$

Since $n_s \approx 1$ up to slow-roll parameters, it follows that the power spectrum is (nearly) scale-invariant. This was a significant prediction of inflation that was confirmed by observational data from the COBE and WMAP experiments. Conversely, this also ruled out competing models based on cosmic strings that had been proposed around the same time [48], but that were incompatible with scale-invariance. We note here that, since this is a result that is on solid empirical ground, any alternatives to inflation (some of which we explore in the following chapters) must necessarily also lead to (near) scale-invariance of the spectrum of curvature perturbations.

2.5 Spectrum of Tensor Perturbations – Primordial Gravitational Waves

Inflation predicts that tensor perturbations in the spatial metric

$$\mathrm{d}s^2 = a^2(\tau)\left(-\mathrm{d}\tau^2 + (\delta_{ij} + h_{ij})\,\mathrm{d}x^i\,\mathrm{d}x^j\right) \tag{2.5.1}$$

should result in the generation of primordial gravitational waves. Recall that the perturbations satisfy $\partial_i h_{ij} = 0$, $h^i_i = 0$. If we expand the Einstein-Hilbert action to second order in the perturbations [38, 46], we obtain

$$S_{(2)} = \frac{M_{pl}^2}{8} \int \mathrm{d}^3\mathbf{x}\, \mathrm{d}\tau\, a^2 \left[\left(h'_{ij} \right)^2 - (\partial_l h_{ij})^2 \right], \tag{2.5.2}$$

where, in comparison to the previous section, we have re-introduced M_{pl}^2 to ensure that h_{ij} is explicitly dimensionless. We see here that this is equivalent to the action for a massless scalar field in an FLRW universe, up to a normalisation factor. As before, we introduce the Fourier transform of the perturbations

$$h_{ij} = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \sum_{s=+,\times} \epsilon^s_{ij}(k)\, h^s_{\mathbf{k}}(\tau) e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{2.5.3}$$

where $\epsilon_{ii} = k^i \epsilon_{ij} = 0$, $\epsilon^s_{ij}(k)\, \epsilon^{\tilde{s}}_{ij}(k) = 2\delta_{s\tilde{s}}$, and $+$, $\times$ are the two polarisation modes of the gravitational waves. Then the action (2.5.2) becomes

$$S_{(2)} = \frac{M_{pl}^2}{4} \sum_{s=+,\times} \int \mathrm{d}\tau\, \mathrm{d}^3\mathbf{k}\, a^2 \left[h^{s'}_{\mathbf{k}} h^{s'}_{\mathbf{k}} - k^2 h^s_{\mathbf{k}} h^s_{\mathbf{k}} \right] \tag{2.5.4}$$

By defining the (canonically normalised) field

$$v^s_{\mathbf{k}} \equiv \frac{a}{2} M_{pl}\, h^s_{\mathbf{k}}, \tag{2.5.5}$$

we find that

$$S_{(2)} = \frac{1}{2} \sum_{s=+,\times} \int \mathrm{d}\tau\, \mathrm{d}^3\mathbf{k} \left[\left(v^{s'}_{\mathbf{k}} \right)^2 - \left(k^2 - \frac{a''}{a} \right) (v^s_{\mathbf{k}})^2 \right] \tag{2.5.6}$$

By comparing to (2.4.18), we are able to recognise this as essentially being two copies of the action for the (canonically normalised) scalar. Varying this as usual, the equations of motion for the mode functions are

$$v^{s''}_k + \left(k^2 - \frac{a''}{a} \right) v^s_k = 0, \tag{2.5.7}$$

where the index s runs over the polarisation modes $+$, $\times$. Using the same method as before, we find that

$$\frac{a''}{a} = \frac{\nu^2 - 1/4}{\tau^2}, \tag{2.5.8}$$

where, in this case, $\nu \approx \frac{3}{2} + \varepsilon$ to first order in the slow-roll approximation and the Bunch-Davies vacuum is the same as before. Taking into account the two polarisations, the power spectrum of tensor fluctuations in the late-time limit is given by

$$\begin{aligned}\Delta_t^2(k) = 2 \cdot \Delta_h^2(k) = 2 \cdot \left(\frac{2}{a\, M_{pl}}\right)^2 \lim_{k\tau \to 0} \frac{k^3}{2\pi^2} \left|v_k^s(\tau)\right|^2 \\ = \frac{2}{\pi^2} \frac{H_*^2}{M_{pl}^2} \left(\frac{k}{k_*}\right)^{3-2\nu} = A_t \left(\frac{k}{k_*}\right)^{3-2\nu},\end{aligned} \tag{2.5.9}$$

where A_t is the amplitude of the spectrum and the subscript $*$ has the same meaning as before. The tensor spectral index is defined as

$$\begin{aligned}n_t \equiv \frac{\mathrm{d}\ln \Delta_h^2}{\mathrm{d}\ln k} = \frac{\mathrm{d}\ln \Delta_h^2}{\mathrm{d}\ln (aH)} \\ \approx \frac{\mathrm{d}\ln \Delta_h^2}{\mathrm{d}\ln a} = \frac{\mathrm{d}\ln \Delta_h^2}{H\, \mathrm{d}t} = 2\frac{\dot{H}}{H^2} = -2\varepsilon\end{aligned} \tag{2.5.10}$$

Using the definition of ν, the exponent in (2.5.9) becomes $3 - 2\nu = -2\varepsilon = n_t$. Therefore, the power spectrum of tensor perturbations also follows a power law

$$\Delta_t^2(k) = A_t \left(\frac{k}{k_*}\right)^{n_t} \tag{2.5.11}$$

As we shall see when making a comparison to cyclic models in Chapter 3, the prediction of primordial gravitational waves is something of a smoking gun for inflation. In terms of relating theoretical predictions to observations, the tensor-to-scalar ratio is usually taken

$$r \equiv \frac{\Delta_t^2(k)}{\Delta_{\mathcal{R}}^2(k)} = \frac{A_t}{A_s} = 16\varepsilon \tag{2.5.12}$$

Using the direct relation between the tensor spectral index and the slow-roll parameter ε (2.5.10), we find the consistency relation for single-field slow-roll inflation models

$$r = -8n_t \tag{2.5.13}$$

Strictly speaking, this should be derived after relating the slow-roll parameters with respect to the potential V to those with respect to the Hubble parameter (which we used here), but the final consistency relation obtained is the same (for details, see e.g. [38]).

2.6 Embedding the Model within a Fundamental Theory – Brane Inflation

While inflation provides a plausible explanation for the horizon, large-scale structure and flatness problems of modern standard cosmology, it assumes the presence of a scalar field in a (flat) potential within its setup. It does not actually explain its origin from the perspective of a fundamental theory. To do this, we require input from a (possibly UV-complete) theory. As mentioned in the introduction, this is especially significant since chaotic inflation is a large-field picture, which prevents its interpretation as an effective field theory at lower energies. A prime candidate for this approach is string theory – specifically, we will consider M-theory, which has been postulated as the overarching description that unifies the 5 major superstring theories: $\mathcal{N} = 1$ Type I, $\mathcal{N} = 2$ Type IIA and IIB, $SO(32)$ and $E_8 \times E_8$ (the latter two are heterotic, i.e. theories of closed strings where the left-moving excitations of the string are bosonic in 26 dimensions and the right-moving ones are superstrings in 10 dimensions [49]). A rigorous treatment of this subject is beyond the scope of this dissertation (and would perhaps entail a thesis in itself); our approach will be to present the construction by which the 4-dimensional visible Universe is embedded within the 11-dimensional theory, as well as a possible means by which inflation may arise in this theory. This section may appear somewhat heuristic, but we hope to impart to the reader a 'visualisation' of the underlying mechanism, and direct his/her attention to the references for further details. In particular, we will follow the path laid out in [17, 18, 19].

Hořava and Witten showed in [50, 51] that the strongly coupled $E_8 \times E_8$ heterotic string coincides with 11-dimensional supergravity; specifically, the $D = 11$ limit of M-theory which is compactified on an $S^1/\mathbb{Z}_2$ orbifold, with E_8 gauge fields living on each of the two $D = 10$ orbifold fixed planes. Furthermore, Witten also demonstrated in [52] that this limit of M-theory can be compactified (without introducing phenomenological issues) on a deformed Calabi-Yau (CY) 3-fold, thereby resulting in an $\mathcal{N} = 1$ supersymmetric theory in $D = 4$. It was also found, by matching to the gravitational and Grand Unified Theory (GUT) couplings at tree level, that the orbifold must be larger than the CY radius (which itself is of the order of the $D = 11$ Planck length) by a factor of ~ 10 [18]. This then implies that there exists a regime where the Universe looks 5-dimensional. Upon reduction from $D = 11$ to $D = 5$ on the CY manifold, it is found that flat space is not a solution of the effective theory without also leading to a decompactification of the CY space. This would present questions about why a fifth extended dimension has not been observed at lower energies. However, the vacuum state that does lead to (an acceptable) $\mathcal{N} = 1$ supersymmetric $D = 4$ space is a 3-brane domain wall (with the CY space remaining compact): then the observed $D = 4$ spacetime can be identified with the worldvolume of the 3-brane. The matter and gauge fields are restricted to the 3-brane, while gravity is allowed to propagate in the bulk as well.

Massive representations of an extended SUSY algebra have mass equal to the SUSY central charge Z (BPS states). If we take linear combinations of the original generators, half the algebra reduces to the anticommutator being zero (due to the condition satisfied by definition by a BPS state, $M = \sqrt{2}Z$). Then the other half of the supersymmetries are preserved. Witten's original background in $D = 5$ preserved 4 supercharges, whereas the required number of preserved supercharges in $\mathcal{N} = 1$, $D = 5$ minimal supergravity is 8. Hence, finding Witten's background (which was derived only to the first non-trivial order in an expansion of Newton's constant in 11 dimensions) as a particular solution of the effective $D = 5$ bulk theory of [17] means that deformations of the CY background (corresponding to Witten's theory) should break half the supersymmetries and preserve the other half (while also preserving Poincaré invariance in $D = 4$ for obvious reasons), thereby taking the number of preserved supercharges from 8 to 4. As a result, the appropriate exact vacuum solution (i.e. the solution that provides the appropriate background for reduction to $D = 4$) is a BPS state.

One may ask why we cannot simply reduce the theory directly from 11 to 4 dimensions. The reason that a 2-step reduction, first from 11 to 5 and then from 5 to 4 dimensions, is performed is two-fold: the scale of the fifth (orbifold) dimension is larger than that of the Calabi-Yau manifold and while the theory can be reduced from 11 to 5 dimensions *à la* Kaluza-Klein and the massive CY modes can be set to zero without requiring higher-order corrections, the 5- to 4-dimensional reduction involves taking into account and integrating out the non-trivial $D = 5$ modes. This yields higher-order corrections which may produce phenomenological signatures.

Witten's 'deformed' Calabi-Yau solution is interpreted [17] as a collection of 5-branes wrapped on 2-cycles and lying in the fixed orbifold planes. In the $D = 5$ picture, these become 3-branes which span the fixed planes of the compact orbifold. Hence, this background is identified with a pair of parallel 3-branes, with the $D = 4$ observed spacetime corresponding to one of these and a 'hidden' sector corresponding to the other. From the point of view of the bulk, we could have had multiple 3-brane solutions with any number of parallel branes along the x^{11} direction. However, these brane solutions have singularities at the locations of the branes and are hence required to be supported by source terms, for which the immediate choices are boundary actions. Keeping in mind the requirements imposed by cancellation of quantun anomalies, the possible solutions get restricted to those representing just a pair of parallel 3-branes, i.e. the orbifold planes.

The discussion up to this point has revolved around how we may derive a worldvolume corresponding to our $D = 4$ visible Universe from the fundamental $D = 11$ M-theory. We now turn our attention to the cosmology of this solution – recall that this is a vacuum and has no matter/energy content, but will be used as a basis for the expected cosmology that we hope to find from this theory. From [17], the reduction of the theory from 11 to 5 dimensions necessitates the presence of a non-zero field strength, which is a 4-form in the internal Calabi-Yau directions. This leads to a gauged supergravity with a potential term (that was unseen prior to the publication of that paper) and includes boundary potentials which project the bulk scalar field onto the orbifold planes. One point of particular interest is that the presence of boundary sources means that solutions of the theory are inhomogeneous in the orbifold direction (also known as the radion), and so flat space cannot be a solution in $D = 5$.

An ideal cosmological solution [18] that satisfies this requirement would involve the 6-dimensional Calabi-Yau space and the orbifold undergoing some evolution for a short period in time and then settling down to their phenomenological values in the 'static' limit at late times, while the observed extended spatial dimensions continue to expand. Then, in this static limit, the theory is effectively 4-dimensional and realistic solutions would be expected to reduce either to the domain wall solution or some modification of it where the remaining $\mathcal{N} = 1$ supersymmetry is broken. As a result, the expected solutions would depend on the orbifold coordinate as well as on time.

In [53, 54], the authors demonstrated the method by which a general class of cosmological solutions could be derived in superstring and M-theories defined in spacetimes without boundary. This was achieved exchanging the time and radial coordinates, and this approach can be utilised in Hořava-Witten theory as well if the radial coordinate is not oriented in the orbifold direction. However, this method cannot be applied to the fundamental 3-brane domain wall solution since it is undesirable for the time and radial coordinates to be exchanged in the vacuum state. Also, the radial direction is aligned in the orbifold direction. Instead, the domain wall itself is made time-dependent so the solutions depend on both the time and orbifold coordinates. The coupled partial differential equations (PDEs) that result from this can be solved by separation of variables under certain constraints.

The most realistic scenario found in the case of the vacuum solution [18] corresponds to the domain wall with 2 of the 3 moduli being time-dependent. These moduli measure the separation of the two walls of the 3-brane and the size of its worldvolume (i.e. the size of the visible Universe). In this case, starting with the ansätz

$$\begin{gathered} \mathrm{d}s_5^2 = -N(t,y)^2\,\mathrm{d}t^2 + a(t,y)^2\eta_{mn}\,\mathrm{d}x^m\,\mathrm{d}x^n + b(t,y)^2\,\mathrm{d}y^2 \\ V = V(t,y) \end{gathered} \tag{2.6.1}$$

one can use separation of variables to solve the equations of motion

$$\begin{gathered} N(t,y) = n(t)a(y), \quad a(t,y) = \alpha(t)a(y) \\ b(t,y) = \beta(t)b(y), \quad V(t,y) = \gamma(t)V(y) \end{gathered} \tag{2.6.2}$$

which only works if $\beta \propto \gamma$ (one can take $\beta = \gamma$ without loss of generality). The solution obtained is

$$\alpha = A|t-t_0|^p, \;\; \beta = \gamma = B|t-t_0|^q, \tag{2.6.3}$$

where $p = \frac{3}{11}\left(1 \mp \frac{4}{3\sqrt{3}}\right)$, $q = \frac{2}{11}\left(1 \pm 2\sqrt{3}\right)$. Then using the notation $\overrightarrow{\alpha} = (\hat{\alpha}, \hat{\beta}, \hat{\gamma})^T$, where $\alpha = e^{\hat{\alpha}}, \beta = e^{\hat{\beta}}, \gamma = e^{6\hat{\gamma}}$, the authors of [18] find that the conformal Hubble parameter (which they define purely as the time derivative with respect to conformal time) takes the form

$$\overrightarrow{\mathcal{H}} \equiv \frac{\mathrm{d}\overrightarrow{\alpha}}{\mathrm{d}\tau} = \frac{\overrightarrow{p}}{\tau - \tau_0}, \tag{2.6.4}$$

where $\overrightarrow{p} = (p, q, q/6)^T$ and τ_0 is the conformal time corresponding to a future curvature singularity. Then there exist two (conformal) time ranges: $\tau < \tau_0$, identified as the $(-)$ branch and $\tau > \tau_0$, identified as the $(+)$ branch. There are also two options $p_\uparrow$ and $p_\downarrow$ corresponding to the two signs of p and q. In the 5-dimensional as well as in the 4-dimensional Einstein frames, the authors of [18] arrive at the conclusion that, in the $(+)$ branch, taking the choice of $p_\downarrow$, the worldvolume of the 3-brane expands while the orbifold contracts. This is recognised as giving the closest description to inflation. Note, however, that this solution describes a vacuum and hence contains no matter or radiation.

This 'skeleton' is built upon in [19], which introduces boundary inflation as a (potentially realistic) means of actualising inflation within M-theory. The setting is now gravity and a scalar field ϕ (which is the volume modulus of the 6-dimensional internal Calabi-Yau space) with a potential of non-peturbative origin residing in the bulk, and a scalar field ϕ_i with a potential V_{4i} and the other Standard Model gauge and matter fields on each orbifold plane (i.e. the boundaries). In the M-theory description, the boundary scalars arise naturally as scalar partners of the matter fields living on the boundary planes. The 5-dimensional theory then has the geometrical structure $\mathcal{M}_5 = S^1/\mathbb{Z}_2 \times \mathcal{M}_4$, with the two 4-dimensional orbifold planes $\mathcal{M}_4^i$, $i = 1, 2$, as the boundaries. The visible Universe is on one of these planes while a 'hidden' sector is on the other. Boundary inflation in this case means that inflation is driven by the boundary potentials and the scalars ϕ_i are the inflatons, whereas the bulk potential $V(\phi)$ is (close to) zero for all values of the bulk scalar ϕ during inflation. Consider the action presented in [19]

$$S_5 = \frac{1}{2\kappa_5^2}\left\{\int_{\mathcal{M}_5}\sqrt{-g}\left[R - \frac{1}{2}\partial_\alpha\phi\partial^\alpha\phi - U(\phi)\right] - \sum_{i=1}^{2}\int_{\mathcal{M}_4^i}\sqrt{-g}\left[\frac{1}{2}\partial_\mu\phi_i\partial^\mu\phi_i + U_i(\phi_i,\phi)\right]\right\},$$
$$U(\phi) = \frac{1}{3}v^2e^{-2\phi} + V(\phi), \quad U_i(\phi_i,\phi) = \mp 2\sqrt{2}ve^{-\phi} + V_i(\phi_i,\phi), \tag{2.6.5}$$

where κ_5 is 5-dimensional Newton's constant, R is the Ricci scalar, $\alpha = 0, 1, 2, 3, 5$ labels the coordinates for $\mathcal{M}_5$, $\mu = 0, 1, 2, 3$ labels the coordinates for $\mathcal{M}_4$ and $y = x^5$ parameterises the orbifold. In particular, we should note that there is an explicit part of the boundary potential that depends on the projection of the bulk scalar ϕ onto the orbifold planes. The origin of this projection is Hořava-Witten theory, and it supports the 3-brane domain wall solution of the 5-dimensional theory. The height of this 'domain wall potential' is denoted by v.

As noted above, boundary inflation must be inhomogeneous in the extra dimension, and so excites Kaluza-Klein modes, the magnitude of which is determined by the dimensionless parameter

$$\epsilon_i \equiv \frac{V_{4i}\mathfrak{R}}{M_5^3}, \tag{2.6.6}$$

where $\mathfrak{R}$ is the separation of the orbifold planes (or the size of the fifth dimension) and M_5 is the fundamental scale of the 5-dimensional theory. Given this parameter, one can divide the theory into two regimes – for $|\epsilon_i| \ll 1$, the excitations can be described using linearised gravity, whereas the full nonlinear theory must be used for $|\epsilon_i| \gg 1$. Taking a decomposition of the bulk fields $g_{\mu\nu}$ and ϕ into the orbifold average plus an orbifold-dependent variation

$$g_{\alpha\beta} = \bar{g}_{\alpha\beta} + \tilde{g}_{\alpha\beta}, \quad \phi = \bar{\phi} + \tilde{\phi}, \tag{2.6.7}$$

where the averages of the variations in the orbifold direction vanish

$$\langle\tilde{g}_{\alpha\beta}\rangle_5 = 0, \quad \langle\tilde{\phi}\rangle_5 = 0, \tag{2.6.8}$$

one can find boundary-picture solutions to the equations of motion for the variations. Inserting the boundary energy-momentum tensor into these solutions shows that

$$\begin{aligned} \mathrm{d}s_5^2 = (\bar{g}_{\alpha\beta} + \tilde{g}_{\alpha\beta})\mathrm{d}x^\alpha\,\mathrm{d}x^\beta = \left(1 + \frac{1}{3}\tilde{\phi}\right) e^{-\bar{\beta}} g_{4\,\mu\nu}\mathrm{d}x^\mu\,\mathrm{d}x^\nu + \left(1 + \frac{4}{3}\tilde{\phi}\right) e^{2\bar{\beta}}\mathrm{d}y^2, \\ \tilde{\phi} = -2e^{\bar{\beta}-\bar{\phi}}\epsilon_{DW}\left(z - \frac{1}{2}\right), \end{aligned} \tag{2.6.9}$$

where $e^{2\bar{\beta}} = \bar{g}_{55}$ and $z = \frac{y}{\mathfrak{R}} \in [0,1]$ is the normalised orbifold coordinate. Then we see that the size of the corrections is proportional to $\epsilon_{DW} = -v\mathfrak{R}/\sqrt{2}$ and the condition for the linearised regime is identified as $\epsilon_{DW} \ll 1$. Looking similarly at the general linearised solutions for the bulk fields in the boundary picture yields the condition $\epsilon_i \ll 1$ for the linearised regime (see Sections 2, 3 of [19] for details). Since the volume of the internal Calabi-Yau space and the orbifold radius could have taken different values in the early Universe (compared to their present-day values), the authors suggest that there is no way of knowing *a priori* whether inflation took place in the linear or in the nonlinear regime, and so both situations are discussed.

In the linear regime, a generic inflating solution in $D = 4$ is lifted up to one in $D = 5$. Then the 5-dimensional solution represents a pair of inflating domain wall 3-branes with inhomogeneities in the extra (orbifold) dimension caused by the boundary potentials[2]. On the other hand, initial inhomogeneities that are not generated by boundary sources are damped away. In this scenario, the positive tension 3-brane contains the corresponds to the visible sector and recedes away along the the orbifold (i.e. the extra dimension), while the negative tension brane sits at the bulk singularity [55, 56]. However, as shown in [57], the negative tension brane collapses in such a setup. This implies that the 5-dimensional spacetime is not stable during the inflationary era, and the 4-dimensional effective theory breaks down. Hence, we conclude that boundary inflation in the linearised regime suffers from critical problems which prevent its realisation.

In the nonlinear case, the authors of [19] assume that the bulk scalar ϕ is stabilised by its potential at a point with $V(\phi) = 0$ and that the boundary potentials admit the possibility of slow roll for the boundary scalars ϕ_i. Then one can find a solution to the equations of motion by separation of variables. Both boundaries exhibit de Sitter-like expansion, with the Hubble parameters related to the potentials as $H_i \sim |V_{4i}|/M_5^3$. In this case, the physical

[2] This is made explicit by the boundary conditions for the equations of motion, c.f. eqs. (2.26)–(2.28) of [19].

size of the orbifold is constant in time and fixed in terms of the boundary potentials as

$$\mathfrak{R}_{phys} = 12\left(\frac{1}{V_1} + \frac{1}{V_2}\right), \tag{2.6.10}$$

where $V_i = V_{4i}/M_5^3$. For solutions without horizon, $V_{41} + V_{42} < 0$ and signals between boundaries reach in a finite time interval (also, every signal emitted in the bulk will reach one of the boundaries). For solutions with horizon, $V_{4i} > 0$ for $i = 1, 2$ and the two boundaries are causally decoupled.

A second approach is also taken for the nonlinear case in [19], where separation of variables is not assumed. The result of the first approach is then recovered if a particular continuous parameter (which is denoted k) in this general solution is set to zero. For all other values, the solution is non-inflating and involves a collapsing orbifold, so the 5-dimensional spacetime is unstable here as well. There is also a periodic function $p(x^{\pm}) = p(t \pm y)$ that appears in the expression for the scale factor in the general solution, so the only plausible stable configuration that remains is that for which $k = 0$ and $p(x^{\pm})$ does not have a large impact on the scale factor (this can be arranged by ensuring that $p(x^{\pm})$ has a small maximum amplitude, which implies $p(x^{\pm}) \simeq$ const., thereby reducing to the aforementioned separating solution).

However, we are presented with two distinct problems – firstly, the separating solution represents an extremely particular choice of initial conditions, with $k = 0$ or $k \ll 1$ required for inflation to be maintained and the 5-dimensional solution to remain stable. A small perturbation in this parameter could cause the Universe to collapse; it is postulated that this fine-tuning issue is related to the problem of stabilisation of the orbifold, and represents a major roadblock.

Second is the presence of the periodic function $p(x^{\pm})$, which encodes information about the initial inhomogeneity in the orbifold direction. It is then implied that this information survives the inflationary era – this appears to contradict the expectation of the paradigm that inflation should dilute out any initial information. Using boundary inflation as an example, it is therefore clear that, while the mechanism of inflation is able to explain the problems of modern FLRW cosmology, its realisation from a fundamental theory (which is inevitably required), while not entirely unattainable, is certainly far from iron-clad.

2.7 Problems with Inflation and Open Questions

In the preceding sections, we have shown how the paradigm of inflation can be used to solve the problems of hot Big Bang cosmology introduced in Chapter 1. However, much of its appeal rested on the expectation that a period of inflation would be able to take place and result in a picture that matches present-day observations, irrespective of the initial state of the early Universe. We will now discuss the extent to which this is actually true, and present some of the major problems facing inflation today. We will consider the problems that were known to plague the simplest models [41, 58], such as power-law inflation, as well as those that have been brought to light [59, 60, 61] in recent times by experiments such as Planck2013 [62] and Planck2015 [63].

Initial Conditions

A serious problem with the paradigm takes shape in the form of the initial conditions required for inflation to even begin. The original view was that inflation would be initiated for the Universe emerging from the original singularity with generic initial conditions. However, it was realised that the scalar field having large kinetic energy and gradient contributions would prevent this from occurring. The assumption used to resolve this was then that, after the Big Bang, all forms of energy would be of the same order. The potential would then quickly come to dominate over the kinetic terms and slow-roll inflation would hence be made possible.

However, as we shall see below, plateau-like models are now favoured against power-law ones, which means that the potential density of the plateau must have an upper bound, $M_I^4 < 10^{-12} M_{pl}^4$, so that the predicted density fluctuations could match observations. Hence, the kinetic and gradient terms would dominate over the potential and inflation could not have been triggered. The net result of this is that, for inflation to have even started and been a viable explanation, the Universe emerging from the singularity must have contained a region that stayed homogeneous until the beginning of the inflationary phase. By comparing the relative value of the potential with respect to the Planck mass, it follows that the size of this region would have to have been [59]

$$r^3(t_{pl}) \gtrsim \left[a(t_{pl})\int_{t_{pl}}^{t_I}\frac{\mathrm{d}t}{a}\right]^3 \sim \left[\frac{a(t_{pl})H(t_{pl})}{a(t_I)H(t_I)}H^{-1}(t_{pl})\right]^3$$
$$> 10^9\left(\frac{10^{16}\,\mathrm{GeV}}{M_I}\right)^3 H^{-3}(t_{pl}) \tag{2.7.1}$$

Therefore, the Universe would have been required to be exceptionally homogeneous and isotropic in order for inflation to begin. Meanwhile, inflation itself has been set forth as an explanation for the observed homogeneity and isotropy of the Universe today, thus manifesting the conceptual magnitude of the problem.

Fine-Tuning Problems – Fluctuations and Effective Cosmological Constant

In Section 2.2, we showed how inflation explains the origin of large-scale structure through amplification of density fluctuations, which are generated by quantum perturbations in the scalar sector. However, the primordial density fluctuations are required to be of the order of $\delta\rho/\rho \sim 10^{-5}$ [61]. However, using a simple example of power-law inflation with $V(\phi) = \frac{1}{4}\lambda\phi^4$, the mass fluctuations are predicted to be at the order of $10^2\lambda^{1/2}$, which would require $\lambda \sim 10^{-14}$. At present, there seem to be no solid theoretical arguments that can provide a justification for a level of fine-tuning that involves nearly 15 orders of magnitude.

The inflaton scalar in its potential acts as an effective cosmological constant in that it also causes a phase of accelerated expansion of the scale factor. A constraint can be found on the size of this effective constant [41]

$$\frac{\Lambda_{eff}}{M_{pl}^4} \leq 10^{-122}, \tag{2.7.2}$$

which leads to another fine-tuning problem for inflation – that of explaining the extremely small magnitude of Λ_{eff}.

Trans-Planckian Physics Problem

If we look at the (Lorentz-invariant) scalar curvature and extrapolate its evolution backwards in time, it appears to be on Planckian scales at the beginning of inflation [41, 64]. By doing this, we are implicitly assuming that General Relativity and the perturbation theory we use to calculate the spectrum of curvature fluctuations, for example, remain valid to arbitrarily high energies and are not particularly sensitive to UV-level (or Planck-scale) physics. This is a risky assumption to make, especially since the model of chaotic inflation that we used to illuminate the predictions of the paradigm requires the scalar field to take large values as it is, which puts it at loggerheads with the former. Furthermore, it has been shown in [65] that, taking an infinitesimally small de Sitter space as the initial state, and treating the background scale factor and primordial tensor perturbations quantum mechanically (using a path integral setup) leads to a breakdown of quantum field theory in curved spacetime. As a result, the background and (more crucially) the Bunch-Davies mode functions must be input by hand to prevent instability of the fluctuations. Hence, a pre-inflationary phase is required to ensure that the Universe is sufficiently flat, of high enough energy and quantum mechanically stable so that inflation can actually occur and explain current observations. This implies that inflation is certainly sensitive to the quantum initial state of the Universe and new physics is required to provide a full explanation.

Entropy and the Second Law

As shown by Penrose in [58, 66], extrapolating the second law of thermodynamics (i.e. increasing entropy in the Universe) backwards in time leads us to the problem of the distribution of entropy. From the initial state of a Big Bang, we should expect that the total entropy at the time of recombination should be nearly maximal, and equally partitioned between the matter-radiation and gravitational sectors. The latter is quantified by the Weyl tensor $C_{\mu\nu\rho\sigma}$, which gives the curvature due to pure gravitational field, and so should be exceedingly large. On the other hand, the observed pattern shows that the total entropy is much smaller than its maximum allowed value, with the partition being near-maximum in the matter-radiation sector and negligible in the gravitational sector. This means that, contrary to the naïve expectation, the Weyl tensor is infinitesimally small. This problem is made worse by assuming an inflationary mechanism in the early Universe, since the initial state would have had to be dominated by a nearly-uniform field, with all other components being negligible. This reduces the value of the initial entropy even further below the maximum, so the problem of fine-tuning entropy is further exacerbated.

Unlikeliness Problem

Observational data from Planck 2013 [62] and Planck2015 [63] showed that the density perturbations have negligible non-Gaussianity and are nearly scale-invariant ($n_s \approx 1$). Moreover, the spectrum of primordial gravitational waves should correspond to a tensor-to-scalar ratio of $r < 0.1$. These results preclude most of the simple single-field inflationary models, such as power-law inflation (e.g. $V(\phi) = \frac{1}{4}\lambda\phi^4$), and favour plateau-like models (e.g. $V(\phi) = \frac{1}{4}\lambda\left(\phi^2 - \phi_0^2\right)^2$). However, somewhat counter-intuitively, the latter require more fine-tuning, occur for more special initial conditions and generate much less inflation than the former [59, 61]. As a result, the probability that slow-roll down a plateau-like potential led to the current Universe becomes infinitesimally small. This has been termed the unlikeliness problem.

Multiverse Problem

Classical inflation is predicted to come to a smooth end when the scalar field reaches the bottom of its potential. However, quantum perturbations can cause the scalar to be 'kicked' far uphill, where it undergoes more inflation (which amplifies the quantum fluctuations further) – this is called 'eternal inflation' [67] and has been shown to be generic [43]. Since inflation amplifies volume exponentially, volume is then selected as the natural measure but the fact that our Universe is not inflating today (and other regions of space should be) means that the probability of its existence is diminished by a factor of $10^{-10^{55}}$ [59, 61]. Moreover, using volume-weighting as the measure, it is found that the simple models (like power-law inflation) are exponentially favoured over plateau-like models [60], even though the data implies that the likelihood is the other way round. This presents a serious problem for the choice of measure and further augments the multiverse issue.

2.8 Discussion

In this chapter, we have presented the mechanism by which inflation solves the horizon, flatness and large-scale structure problems. We used the dynamics of the scalar field to show how the slow-roll approximation is obtained, and used linear perturbation theory to derive the power spectrum of curvature fluctuations (which are related to the spectrum of scalar parturbations) and tensor perturbations (which represent primordial gravitational waves). We also provided an overview of how inflation could be visualised as fitting into a fundamental M-theory picture, through the construction of boundary inflation. Finally, we presented some of the problems facing the inflationary paradigm today, both those that have been known for 20-30 years and those that were found in light of recent observational data.

With respect to the latter, a shift in thought to 'postmodern inflation' [60, 68] has been proposed, wherein the measure is selected *a posteriori* to fit the observations. This would clearly mean that there is no straightforward means by which the paradigm as a whole can be proven or disproven, which brings along its own set of challenges.

There are still two more issues that have not been presented, since inflation does not attempt to answer them. The first is (as mentioned in Section 2.1) the origin of the Universe and the initial singularity, and the second is the position of dark energy in a 'complete' model of cosmology. In the standard Λ-CDM model, dark energy (represented by Λ) is an *ad hoc* insertion, and the theory would be on just as sure a footing without it. It is through issues like these that we find motivation to search for alternatives to inflation.

Bouncing Cosmology – A Plausible Alternative?

3

In Chapter 2, we presented the means by which inflation resolves the problems of hot Big Bang cosmology, and also discussed the problems it faces as a paradigm. In this chapter, we will show how the latter yields motivation for considering alternative frameworks but we must first make sure that the original problems are indeed resolved by bouncing cosmologies.

All bouncing models immediately solve the horizon problem without difficulty, since coordinate time does not 'begin' at the singularity $t = 0$, but is extended arbitrarily far back into the past, $t \to -\infty$. Hence regions of space which may not seem casually connected at the time of recombination were in fact connected in the contracting phase prior to the bounce.

Resolution of the flatness problem is model-dependent – in the ekpyrotic/cyclic models that most of this chapter is based on, the spatial curvature decreases faster in the contracting phase than it does in the expanding phase [29]. In particular, the scale factor decreases slowly while the magnitude of the Hubble parameter increases rapidly, so (2.1.6) becomes

$$(\Omega(t_f) - 1) \approx (\Omega(t_i) - 1)\frac{H_i^2}{H_f^2}, \tag{3.0.1}$$

where the subscripts i and f refer to the beginning and end of the period of contraction respectively. Then the ratio $\frac{\Omega(t_f)-1}{\Omega(t_i)-1}$ is exceedingly small at the end of contraction (and hence at the bounce), and does not significantly increase through the expanding phase. A similar result is found in the pre-Big Bang model [69]. In the matter bounce scenario [70], the bounce is symmetric so the flatness problem is still resolved since the curvature decreases by the same amount during contraction that it expands by during expansion.

For the large-scale structure formation problem, curvature perturbations in a single-field ekpyrotic/cyclic model produce a spectrum that is strongly tilted towards the blue, so this mechanism is incompatible with observations [71, 72]. A two-field approach employing the isocurvature/entropic mechanism is used instead to generate the required scale-invariant spectrum of curvature perturbations. Note that it is sub-horizon fluctuations generated in the contracting phase that become larger than the horizon as the Universe approaches the bounce, and continue as super-horizon fluctuations in the expanding phase [34]. The matter-bounce model utilises the duality of the canonical fluctuation variables in inflation and in a matter-dominated contracting Universe, so the structure-formation problem is resolved here in broadly the same manner as in inflation, i.e. through perturbations in the scalar field.

The question of the origin of the Universe, insofar as it refers to the problem of initial conditions, is resolved in bouncing cosmologies by imposing the initial conditions as far back in the past as possible, and so this problem is removed from the problem of the singularity at $t = 0$. For the latter, string-based or string-motivated cosmologies still assume that $t = 0$ refers to a singular transition, so a satisfactory UV-complete theory of gravity incorporating Planck-scale physics is needed. However, the actual singularity is made milder in the ekpyrotic model, since it corresponds to a collision of the visible and hidden branes, and a brief collapse of only the orbifold dimension (c.f. Section 2.4), rather than the full space-time experiencing a singularity. On the other hand, models involving non-singular bounces have also been proposed, in which case their validity is not necessarily dependent on a full quantum theory of gravity. We will explore one such model in detail in Chapter 4.

Before we delve into the ekpyrotic/cyclic realisation of a bouncing cosmology, let us consider the motivations for searching for alternatives to inflation. Firstly, bouncing models avoid the problem of trans-Planckian physics since (fluctuations in) the scalar curvature remain(s) at a scale that is multiple orders below the Planck scale [29]. Secondly, the unlikeliness and multiverse problems, as discussed in Chapter 2, represent a departure from predictability and some use the anthropic principle to justify the choice of measure and weighting. This certainly leaves something to be desired from the perspective of science. The predictability of a scientific theory makes it unique, and this should not be sacrificed for the sake of a particular model.

The entropy problem also presents an outstanding issue that is yet to be resolved from the perspective of inflation. By contrast, in the cyclic picture, a natural resolution is found in that causality limits the size of a black hole to the Hubble radius at the time of collision. Relating the entropy generated at collision to the number of Hubble volumes inside the comoving Hubble volume today and the total mass inside the Hubble radius at collision, one finds that [20]

$$\frac{S_{collision}}{S_{max}} < \frac{T_0}{T_{collision}}. \tag{3.0.2}$$

Then, if the collision temperature lies between 10^5 GeV and 10^{19} GeV, the collision entropy has an upper limit of 10^{-20} to 10^{-30} of the maximum possible entropy.

Finally, we have the puzzle of dark energy, which has been introduced into the Λ-CDM model of cosmology after its discovery, but is certainly not required by any stretch of imagination, in order for inflation to work. This leads us to wonder whether we could find an overarching model of cosmological evolution that incorporates the presence of dark energy; we will show in this chapter that it does indeed play a significant role in the ekpyrotic/cyclic model of the Universe.

3.1 Background Dynamics

The cyclic model involves the same theoretical setting as that presented in Section 2.4, i.e. 11-dimensional M-theory compactified on an $S^1/\mathbb{Z}_2$ orbifold, which is reduced to 5 dimensions on the Calabi-Yau manifold. The original ekpyrotic model [21] involved a bulk brane peeling away from the hidden brane and spontaneously moving towards the visible brane, with the resulting collision being identified as the hot Big Bang. The cyclic Universe that this model evolved into, consists of the visible and hidden branes periodically moving towards each other along the orbifold dimension, with the collisions recognised as Big Crunch/Big Bang transitions. As a starting point for ekpyrotic contraction, we consider the action

$$S = \int \mathrm{d}^4x\, \sqrt{-g} \left(\frac{1}{2}R - \frac{1}{2}\partial_\mu\phi\, \partial^\mu\phi - V(\phi) \right) \tag{3.1.1}$$

where, as before, R is the (effective 4-dimensional) Ricci scalar. Noting that a cyclic Universe involves a non-singular transition from expansion to contraction [73], we require that the Hubble parameter must go from positive to negative. This can be realised through a negative potential which cancels the positive kinetic energy of matter. Then one possible form of the potential is

$$V(\phi) = -V_0\, e^{-c\phi}, \tag{3.1.2}$$

where V_0 and c are constants. Using the equation of state (2.3.4), this implies that $w_\phi > 1$. If we consider the generalisation of the first Friedmann equation (2.1.2) to include matter, radiation, anisotropy and scalars

$$H^2 = \frac{1}{3}\left(\frac{-3\kappa}{a^2} + \frac{\rho_m}{a^3} + \frac{\rho_r}{a^4} + \frac{\sigma^2}{a^6} + \cdots + \frac{\rho_\phi}{a^{3(1+w_\phi)}} \right), \tag{3.1.3}$$

where κ is the spatial curvature, ρ_m is the density of matter, ρ_r is the density of radiation and σ^2 is the density of anistropies in the curvature, then we find that $w > 1$ implies that the scalar will dominate the evolution in a contracting phase. The equations of motion that result from varying (3.1.1) are

$$H^2 = \frac{1}{3}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right] \tag{3.1.4}$$

$$\frac{\ddot{a}}{a} = -\frac{1}{3}\left[\dot{\phi}^2 - V(\phi)\right] \tag{3.1.5}$$

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial \phi} = 0 \tag{3.1.6}$$

One relation that will prove useful is

$$\dot{H} = -\frac{1}{2}\,\dot{\phi}^2 \tag{3.1.7}$$

If $c \gg 1$, then we obtain the following scaling solution

$$a(t) = (-t)^p, \;\; \phi = \frac{2}{c}\ln\left(-\sqrt{\frac{c^2 V_0}{2t}}\right), \;\; p = \frac{2}{c^2} \tag{3.1.8}$$

Hence, a slowly contracting Universe has the equation of state

$$w = \frac{2}{3p} - 1 \gg 1 \tag{3.1.9}$$

In fact, the condition (3.1.9) can be taken to define the ekpyrotic phase.

The cyclic Universe uses a slight modification of (3.1.1) [22]

$$S = \int \mathrm{d}^4x\,\sqrt{-g}\left(\frac{1}{2}R - \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - V(\phi) + \beta^4(\phi)(\rho_m + \rho_r)\right), \tag{3.1.10}$$

where $\beta(\phi)$ represents the coupling between the scalar ϕ and matter/radiation. This is especially significant as it ensures that the densities ρ_m, ρ_r are finite at the bounce. The equations of motion that we obtain are

$$H^2 = \frac{1}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi) + \beta^4\rho_m + \beta^4\rho_r\right) \tag{3.1.11}$$

$$\frac{\ddot{a}}{a} = -\frac{1}{3}\left(\dot{\phi}^2 - V(\phi) + \frac{1}{2}\beta^4\rho_m + \beta^4\rho_r\right) \tag{3.1.12}$$

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial \phi} + \frac{\partial \beta}{\partial \phi}\,\beta^3 \rho_m = 0 \tag{3.1.13}$$

For matter or radiation, the fluid equation of motion that we obtain is

$$\hat{a}\,\frac{\partial \rho_i}{\partial \hat{\rho}} = a\,\frac{\partial \rho_i}{\partial a} + \frac{\beta}{\beta_{,\phi}}\frac{\partial \rho_i}{\partial \phi} = -3(\rho_i + P_i), \tag{3.1.14}$$

where $\hat{a} = a\beta(\phi)$, i represents matter or radiation and P_i is the pressure of the fluid with density ρ_i. The potential is required to satisfy the following properties

- $V(\phi)$ rapidly as $\phi \to -\infty$ at the bounce
- $V(\phi) < 0$ for intermediate ϕ, i.e. for ekpyrotic contraction
- As ϕ increases, V rises to a shallow plateau with $V_0 > 0$, representing the current period of accelerated expansion

To ensure this, we take the exemplar potential

$$V(\phi) = V_0\left(1 - e^{-c\phi}\right)\,F(\phi), \tag{3.1.15}$$

where $F(\phi)$ is used to ensure compatibility with the first condition, and can be taken as unity to the right of the potential minimum without loss of generality. We also require a matching rule to connect from the Big Crunch to the Big Bang. Since the brane collision is predicted to be partially inelastic, some fraction of the kinetic energy of the scalar field is converted to matter and radiation, and production of a finite density of particles occurs [74, 75]. The rule is

$$\dot{\phi}\,e^{\sqrt{3/2}\phi} \to -(1+\chi)\,\dot{\phi}\,e^{\sqrt{3/2}\phi}, \tag{3.1.16}$$

where the parameter χ measures the efficiency of radiation production at the bounce.

At this point, let us consider the picture of cosmological evolution in the cyclic Universe [22]. We start our journey in the present epoch, when the radiation- and matter-dominated eras have concluded. The potential energy dominates, causing a period of accelerated expansion that lasts for hundreds of e-folds. Note here that one can attribute the density of dark energy to the value of V_0 in the potential, rather than simply adding the former as an *ad hoc* insertion (as in the consensus picture). Thus, the scalar field in its potential $V(\phi)$ provides motivation for dark energy as well as the reasoning for the the Universe being homogeneous, isotropic and flat before the bounce. The potential causes the scalar ϕ to roll in the negative direction, and this stage continues until ϕ reaches the point of zero potential energy. At this point, the kinetic energy of the scalar dominates but the expansion damps its motion. Ultimately, we reach a point when the total energy is zero; using the equations of motion (3.1.11) and (3.1.12), the Hubble parameter is briefly zero and the scale factor is contracting, $\ddot{a} < 0$. As gravitational energy is converted into kinetic energy of ϕ, the field goes past the minimum of its potential and continues on towards $-\infty$, which represents the moment of the Big Crunch/Big Bang transition. During this time, the Universe is increasingly dominated by scalar kinetic energy. After the bounce, radiation is produced and expansion begins again, thereby completing the cycle. Note that, in the kinetic-energy dominated phase right before the bounce, the equations of motion become [73]

$$3H^2 = \frac{1}{2}\dot{\phi}^2 = -\dot{H}, \qquad \ddot{\phi} + 3H\dot{\phi} = 0, \tag{3.1.17}$$

which yield $a \propto e^{\phi/\sqrt{6}}$ and are solved by

$$a = a_0(-t)^{1/3}, \qquad \phi = \sqrt{\frac{2}{3}}\ln(-t) + \phi_0, \tag{3.1.18}$$

where a_0 and ϕ_0 are integration constants.

However, the single field mechanism in this cyclic model generates a power spectrum of curvature perturbations that is not scale-invariant. Then, foreshadowing the fact that this model will require two scalar fields in order to generate the correct spectrum through the entropic/isocurvature mechanism (which we discuss in the next section), we identify these as the radion field that determines the distance between the branes along the orbifold dimension and the volume modulus of the internal Calabi-Yau manifold [73]. However, this presents another issue – the classical trajectory followed by the two scalar fields down the potential ends up being unstable to quantum fluctuations in the transverse direction [76] and the majority of the Universe does not survive. However, if the period of accelerated expansion dominated by dark energy lasts at least 600 billion years, then a sufficiently large region survives the evolution and completes the cycle [77]. This region grows in size and corresponds to the visible Universe today (in the present cycle, for example). The repeated 'destruction' and 'rebirth' of the Universe from the ashes prompted Lehners and Steinhardt to term this version of the cyclic model 'the phoenix Universe', in a nod to the mythical creature of lore (and recalling somewhat of a parallel to Lemaître's construction [78]). At this point, note that the role of dark energy becomes significant, in that it ensures the evolution of the Universe completes the cycle in addition to making it sufficiently homogeneous, isotropic and flat.

For the phoenix Universe, we can use the same action (3.1.10), generalised to 2 scalars so that

$$S = \int \mathrm{d}^4x \sqrt{-g} \left(\frac{1}{2}R - \frac{1}{2}\sum_{i=1}^{2}(\partial\phi_i)^2 - V(\phi_1, \phi_2) \right) \tag{3.1.19}$$

Then the equations of motion that we obtain are

$$3H^2 = \frac{1}{2}\left(\dot{\phi}_1^2 + \dot{\phi}_2^2\right) + V(\phi_i) + \beta_+^4(\phi_i)\,\rho_+ + \beta_-^4(\phi_i)\,\rho_- \tag{3.1.20}$$

$$3\frac{\ddot{a}}{a} = -\left(\dot{\phi}_1^2 + \dot{\phi}_2^2\right) + V(\phi_i) - \beta_+^4(\phi_i)\,\rho_+ - \beta_-^4(\phi_i)\,\rho_- \tag{3.1.21}$$

$$\ddot{\phi}_i + 3H\dot{\phi}_i + \frac{\partial V}{\partial \phi_i} = 0, \tag{3.1.22}$$

where the relevant degrees of freedom aside from the scalars are radiation with densities $\rho_\pm$ on the positive/negative tension branes. Recall our useful relation (3.1.7), which becomes

$$\dot{H} = -\frac{1}{2}\sum_{i=1}^{2}\dot{\phi}_i^2 \tag{3.1.23}$$

The equations of motion form a closed system, subject to the continuity equation

$$\frac{\partial}{\partial t}\left(\beta_\pm^4\rho_\pm\right) + 4H\beta_\pm^4\rho_\pm = 0 \rightarrow \beta_+^4\rho_+ + \beta_-^4\rho_- = \frac{\rho_0}{a^4} \tag{3.1.24}$$

This implies that all radiation can be treated as one. The potential in the effective 4-dimensional theory is $V = V_{ek}(\phi_1, \phi_2) + V_{rep}(\phi_2)$, where the repulsive part V_{rep} emphasises the existence of a boundary to scalar field space at $\phi_2 = 0$. This corresponds to the negative tension brane being repelled by a naked singularity in the bulk (in the higher-dimensional theory). Following [77], we can take $V_{rep} \propto \phi_2^{-2}$. Meanwhile, the ekpyrotic part of the potential takes the generalised form of the potential presented earlier

$$V_{ek} = -V_1\, e^{-c_1\phi_1} - V_2\, e^{-c_2\phi_2} + V_0 \tag{3.1.25}$$

In the ekpyrotic phase, the background trajectory follows the scaling solution

$$a = (-t)^{1/\epsilon}, \qquad \phi_i = \frac{2}{c_i}\ln\left(-\sqrt{\frac{c_i^2 V_i}{2t}}\right), \tag{3.1.26}$$

where $\frac{1}{\epsilon} = \frac{2}{c_1^2} + \frac{2}{c_2^2} \ll 1$ is the equivalent of the slow-roll parameter ε for inflation. In particular, the parameter ϵ here is related to the equation of state as $\epsilon = \frac{3}{2}(1+w) \gg 1$.

It will be natural to consider the dynamics of the background in terms of the combinations of scalars in the parallel (σ) and transverse (s) directions, particularly as curvature perturbations in the latter will yield the scale-invariant power spectrum that we require [79]. The parallel and transverse combinations of the scalars are

$$\sigma = \frac{\dot{\phi}_1 \phi_1 + \dot{\phi}_2 \phi_2}{\dot{\sigma}}, \qquad s = \frac{\dot{\phi}_1 \phi_2 - \dot{\phi}_2 \phi_1}{\dot{\sigma}}, \tag{3.1.27}$$

where $\dot{\sigma} = \left(\dot{\phi}_1^2 + \dot{\phi}_2^2\right)^{1/2}$. The angle of the trajectory followed by the scalars in field space is given by

$$\cos\theta = \frac{\dot{\phi}_1}{\dot{\sigma}}, \qquad \sin\theta = \frac{\dot{\phi}_2}{\dot{\sigma}} \tag{3.1.28}$$

Then the potential becomes

$$V = -V_0\, e^{\sqrt{2\epsilon}\sigma} \left[1 + \epsilon s^2 + \frac{\kappa_3}{3!}\epsilon^{3/2} s^3 + \frac{\kappa^4}{4!}\epsilon^2 s^4 + \dots\right] \tag{3.1.29}$$

If the ekpyrotic part of the potential has the exact form (3.1.25), then

$$\kappa_3 = \frac{2\sqrt{2}\,(c_1^2 - c_2^2)}{|c_1 c_2|}, \qquad \kappa_4 = \frac{4\,(c_1^6 + c_2^6)}{c_1^2 c_2^2\,(c_1^2 + c_2^2)} \tag{3.1.30}$$

The classical ekpyrotic trajectory is stable through σ and unstable through $s = 0$; this implies that there is a ridge at $s = 0$, the evolution along which is given by the scaling solution

$$a(t) = (-t)^{1/\epsilon}, \qquad \sigma = -\sqrt{\frac{2}{\epsilon}} \ln\left(-\sqrt{\epsilon V_0}t\right), \qquad s = 0 \tag{3.1.31}$$

3.2 Spectrum of Curvature Perturbations

We now consider the generation of a nearly scale-invariant spectrum of curvature perturbations through the entropic/isocurvature mechanism. To ensure that the spectrum is *approximately* (rather than exactly) scale-invariant, and hence compatible with observations, we take the potential (3.1.25) as [80]

$$V_{ek} = -V_1\, e^{-\int c_1\, \mathrm{d}\phi_1} - V_2\, e^{-\int c_2\, \mathrm{d}\phi_2}, \tag{3.2.1}$$

where $c_1 = c_1(\phi_1)$, $c_2 = c_2(\phi_2)$ and $V_1, V_2 > 0$. In the conformal Newtonian gauge, scalar perturbations of a flat FLRW metric are given by

$$\mathrm{d}s^2 = -(1+2\Phi)\,\mathrm{d}t^2 + a^2(t)(1-2\Psi)\,\mathrm{d}\mathbf{x}^2 \tag{3.2.2}$$

and the lack of anisotropic stress means that $\Phi = \Psi$. The most general gauge transformation that preserves this metric (up to spatial gradients) is

$$\delta t = \frac{\alpha_1(\mathbf{x})}{a} - \frac{\alpha_2(\mathbf{x})}{a}\int^t \mathrm{d}t'\,a(t'), \qquad \delta x^i = \alpha_2(\mathbf{x})\,x^i, \tag{3.2.3}$$

where the constant coefficients are replaced by the functions α_1 (a local time delay to/from the singularity) and α_2 (a local dilatation or curvature perturbation). Then the Newtonian potential becomes

$$\Phi = \alpha_1(\mathbf{x})\frac{\dot{a}}{a^2} + \alpha_2(\mathbf{x})\left(1 - \frac{\dot{a}}{a^2}\int^t \mathrm{d}t'\,a(t')\right) \tag{3.2.4}$$

In an expanding Universe, the first term is decaying and the second term is growing, and vice versa in a contracting Universe.

In particular, we will consider the entropy perturbation, which is the relative fluctuation in the fields given by

$$\delta s \equiv \frac{\dot{\phi}_1\delta\phi_2 - \dot{\phi}_2\delta\phi_1}{\sqrt{\dot{\phi}_1^2 + \dot{\phi}_2^2}} \tag{3.2.5}$$

Under a linearised coordinate transformation, this is a gauge-invariant quantity so we expect that it should remain unscathed with only Planck-suppressed corrections, even in the presence of gravity. Then the entropic perturbation equation is [81]

$$\delta\ddot{s} + 3H\delta\dot{s} + \left(\frac{k^2}{a^2} + V_{ss} + 3\dot{\theta}^2\right)\delta s = \frac{4k^2\dot{\theta}}{a^2\sqrt{\dot{\phi}_1^2 + \dot{\phi}_2^2}}, \tag{3.2.6}$$

where

$$V_{ss} = \frac{\dot{\phi}_2^2 V_{,\phi_1\phi_1} - 2\dot{\phi}_1\dot{\phi}_2 V_{,\phi_1\phi_2} + \dot{\phi}_1^2 V_{,\phi_2\phi_2}}{\dot{\phi}_1^2 + \dot{\phi}_2^2} \tag{3.2.7}$$

and

$$\dot{\theta} = \frac{\dot{\phi}_2 V_{,\phi_1} - \dot{\phi}_1 V_{,\phi_2}}{\dot{\phi}_1^2 + \dot{\phi}_2^2} \tag{3.2.8}$$

Since we will focus only on straight-line trajectories in the scalar field space, $\dot{\theta} = 0$ so the right-hand side of (3.2.6) is equal to zero. We assume that the background solution obeys scaling symmetry, so that

$$\dot{\phi}_2 = \gamma \dot{\phi}_1, \qquad \phi_1 = \phi, \tag{3.2.9}$$

where γ is a constant. Let us consider the rescaled entropy field $\delta S = a(\tau)\, \delta s$, where τ is (as before) the conformal time, so that (3.2.6) becomes

$$\delta S'' + \left(k^2 - \frac{a''}{a} + a^2 V_{,\phi\phi} \right) \delta S = 0 \tag{3.2.10}$$

As before, primes denote derivatives with respect to conformal time. At this point, we can make use of our previous calculation for inflation from Section 2.2. Specifically, we write the significant second term of (3.2.10) in a manner suggestive of that in (2.4.30)

$$\tau^2 \left(\frac{a''}{a} - a^2 V_{,\phi\phi} \right) = \nu_s^2 - \frac{1}{4} \tag{3.2.11}$$

When this term is approximately equal to 2, we obtain nearly scale-invariant perturbations. We recall the parameter ϵ as

$$\epsilon \equiv \frac{3}{2}(1+w) \equiv \frac{\dot{\phi}_1^2 + \dot{\phi}_2^2}{2H^2} = \frac{(1+\gamma^2)\dot{\phi}^2}{2H^2} = -\frac{\dot{H}}{H^2}, \tag{3.2.12}$$

where we have used (3.1.23) to make the connection to the last equality. In the background scaling solution, we have that $\epsilon = \frac{c^2}{2(1+\gamma^2)}$.

We now expand (3.2.11) in inverse powers of ϵ and its derivatives with respect to $N = \ln(a/a_{end})$, where a_{end} is the value of the scale factor at the end of ekpyrosis. N decreases as the fields roll downhill and the contraction progresses. Differentiating the expression (3.1.23), the first term is

$$\frac{a''}{a} = 2H^2 a^2 \left(1 - \frac{1}{2}\epsilon \right) \tag{3.2.13}$$

We obtain the second term by differentiating ϵ twice with respect to the coordinate time t, and using the background equations and the definition of N to yield

$$a^2 V_{,\phi\phi} = -a^2 H^2 \left(2\epsilon^2 - 6\epsilon - \frac{5}{2}\epsilon_{,N} \right) + \mathcal{O}(\epsilon^2) \tag{3.2.14}$$

Now we require the conformal Hubble parameter $\mathcal{H} = \frac{a'}{a} = aH$ in terms of conformal time. From (3.2.13), we obtain $\mathcal{H}' = \mathcal{H}^2(1-\epsilon)$. Integrating this yields[1]

$$\mathcal{H}^{-1} = \int_0^{\tilde{\tau}} \mathrm{d}\tau\,(\epsilon - 1) \tag{3.2.15}$$

Then taking $1 = \frac{\mathrm{d}\tau}{\mathrm{d}\tau}$ and integrating by parts,

$$\mathcal{H}^{-1} = \epsilon\tau\left(1 - \frac{1}{\epsilon} - (\epsilon\tau)^{-1}\int_0^{\tilde{\tau}} \mathrm{d}\tau\,\epsilon'\tau\right) \tag{3.2.16}$$

Repeating this trick, we obtain

$$(\epsilon\tau)^{-1}\int_0^{\tilde{\tau}} \mathrm{d}\tau\,\epsilon'\tau = \frac{\epsilon'\tau}{\epsilon} - (\epsilon\tau)^{-1}\int_0^{\tilde{\tau}} \mathrm{d}\tau\,\frac{\mathrm{d}}{\mathrm{d}\tau}(\epsilon'\tau)\tau \tag{3.2.17}$$

Using $\epsilon' = \mathcal{H}\epsilon_{,N}$ and the fact that, to leading order in $1/\epsilon$, $\mathcal{H}$ can be replaced by $\mathcal{H}\tau = \epsilon^{-1}$ (from the scaling solution), the second term on the right-hand side of (3.2.17) becomes

$$-(\epsilon\tau)^{-1}\int_0^{\tilde{\tau}} \mathrm{d}\tau\,\frac{\mathrm{d}}{\mathrm{d}\tau}(\epsilon'\tau)\tau = (-\epsilon\tau)^{-1}\int_0^{\tilde{\tau}} \mathrm{d}\tau\,\frac{1}{\epsilon}\left(\frac{\epsilon_{,N}}{\epsilon}\right)_{,N} \tag{3.2.18}$$

Then this term is of the order $\mathcal{O}(1/\epsilon^2)$, and so can be neglected. Hence,

$$\mathcal{H}^{-1} = \int_0^{\tilde{\tau}} \mathrm{d}\tau\,(\epsilon - 1) \approx \epsilon\tau\left(1 - \frac{1}{\epsilon} - \frac{\epsilon_{,N}}{\epsilon^2}\right) \tag{3.2.19}$$

Finally, (3.2.11) becomes

$$\tau^2\left(\frac{a''}{a} - a^2 V_{,\phi\phi}\right) = 2\left(1 - \frac{3}{2\epsilon} + \frac{3}{4}\frac{\epsilon_{,N}}{\epsilon^2}\right) = \nu_s^2 - \frac{1}{4} \tag{3.2.20}$$

which implies that

$$\nu_s \approx \frac{3}{2} - \frac{3}{\epsilon} + \frac{1}{2}\frac{\epsilon_{,N}}{\epsilon^2} \tag{3.2.21}$$

Using the same line of reasoning as in Section 2.2, we find that the spectral index is given by

$$n_s - 1 = \frac{2}{\epsilon} - \frac{\epsilon_{,N}}{\epsilon^2} \tag{3.2.22}$$

[1] To prevent confusion between the integration variable and the limit of integration, we take a tilde on the latter and then replace $\tilde{\tau} \to \tau$ after the integration.

However, the story is only half-complete at this stage; we need to find the power spectrum of curvature perturbations, and so need a mechanism to convert the entropy perturbations into curvature ones. To do this, we follow the method outlined in [80, 81]. Taking (as for inflation) $\mathcal{R}$ to be the curvature perturbation on comoving spatial slices, for N scalar fields with general Kähler metric $g_{ij}(\phi)$ in scalar field space, the linearised Einstein-scalar field equations yield

$$\dot{\mathcal{R}} = -\frac{H}{\dot{H}}\left(g_{ij}\frac{\mathrm{D}^2\phi^i}{\mathrm{D}t^2}s^j - \frac{k^2}{a^2}\Psi\right) \tag{3.2.23}$$

where the $(N-1)$ entropy perturbations

$$s^i = \delta\phi^i - \dot{\phi}^i\frac{g_{jk}(\phi)\dot{\phi}^j\delta\phi^k}{g_{lm}(\phi)\dot{\phi}^l\dot{\phi}^m} \tag{3.2.24}$$

are the transverse components of $\delta\phi^i$ with respect to the backgroiund trajectory, and $\frac{\mathrm{D}^2}{\mathrm{D}t^2}$ is the geodesic operator on scalar field space. In the scenario we have under consideration, the scalar field space is flat so $g_{ij} = \delta_{ij}$ and $\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t}$. Since we have two scalar fields,

$$s^1 = -\frac{\dot{\phi}_2\,\delta s}{\sqrt{\dot{\phi}_1^2+\dot{\phi}_2^2}}, \qquad s^2 = +\frac{\dot{\phi}_2\,\delta s}{\sqrt{\dot{\phi}_1^2+\dot{\phi}_2^2}} \tag{3.2.25}$$

Using a straight line trajectory in field space, the right-hand side of (3.2.24) vanishes even if the entropy perturbations are non-zero. If the trajectory differs from geodesic motion, the entropy perturbation directly yields the curvature perturbation. This occurs when scalar potentials are present, and is expected to happen in general when the ekpyrotic potentials switch off. The bending of the scalar field trajectory that is caused is dependent on the actual model chosen.

However, if say ϕ_2 reflects off a boundary (at $\phi_2 = 0$, for example) at some time t_b, then we have a model-independent contribution. Assume the bounce occurs after the ekpyrotic potentials turn off, so the Universe is dominated by the kinetic energy of the scalar field in the 4-dimensional effective theory. The scalar field trajectory is

$$\dot{\phi}_2 = \begin{cases} -\tilde{\gamma}\dot{\phi}_1, & t < t_b \\ +\tilde{\gamma}\dot{\phi}_1, & t > t_b \end{cases}, \tag{3.2.26}$$

where $\dot{\phi}_1 = \text{const.} < 0$ around the point of the bounce. The bounce itself results in a delta function on the right-hand side of (3.2.23),

$$\frac{\mathrm{D}^2\phi_2}{\mathrm{D}t^2} = \delta(t-t_b)\,2\dot{\phi}_2(t_b^+) \tag{3.2.27}$$

If the spectrum of entropic perturbations is already scale-invariant by the time t_b, then the bounce results in an immediate conversion into curvature perturbations with the same spectrum at long wavelengths. The ekpyrotic contracting phase ends at a time t_{end}, which corresponds approximately to $|V_{min}| = \frac{2}{c^2 t_{end}^2}$. After t_{end}, δs obeys

$$\delta\ddot{s} + \frac{1}{t}\,\delta\dot{s} = 0 \rightarrow \delta s = A + B\ln(-t) \tag{3.2.28}$$

Matching this to the growing mode solution t^{-1} in the contracting phase, the entropy grows by an extra factor of $1 + \ln(t_{end}/t_b)$ by the time of the bounce. By making use of the first Friedmann equation to relate $\dot{\phi}_2 = \tilde{\gamma}\dot{\phi}_1$ to H and explicitly restoring M_{pl}, we find that

$$\langle \mathcal{R}^2 \rangle = \frac{c^2|V_{min}|}{3\pi^2 M_{pl}^2} \frac{\tilde{\gamma}^2}{(1+\tilde{\gamma}^2)^2} \left(1 + \ln\left(\frac{t_{end}}{t_b}\right)\right)^2 \int \frac{\mathrm{d}k}{k} \equiv \int \frac{\mathrm{d}k}{k}\,\Delta^2_{\mathcal{R}}(k) \tag{3.2.29}$$

which yields the amplitude of the power spectrum of curvature perturbations, as required. Specifically, the observed value of $\Delta^2_{\mathcal{R}}(k)$ constrains $c\,|V_{min}|^{1/2} \approx 10^{-3}M_{pl}$, which is around the GUT scale, and therefore reconcilable with our expectation from the underlying heterotic M-theory picture.

3.3 Spectrum of Tensor Perturbations – Departure from the Predictions of Inflation

In this section, we will encounter our first departure from the predictions of the inflationary paradigm. Tensor perturbations in the ekpyrotic/cyclic model can be analysed in a similar manner to those in inflation, but will be found to imply a power spectrum of gravitational waves which are negligible on the scales of the the present time [21, 73, 82, 83]. The perturbed metric is

$$\mathrm{d}s^2 = a^2\left(-\mathrm{d}\tau^2 + \left(\delta_{ij} + 2h_T Y^{(2)}_{ij}\right)\right)\mathrm{d}x^i\,\mathrm{d}x^j, \tag{3.3.1}$$

where $Y^{(2)}_{ij}$ is a tensor harmonic. The tensor perturbation is gauge-invariant, and satisfies the equation of motion

$$h_T'' + 2\frac{a'}{a}h_T' + k^2 h_T = 0 \tag{3.3.2}$$

Using the rescaled variable $f_T \equiv a h_T$, this becomes

$$f_T'' + \left(k^2 - \frac{a''}{a}\right) f_T = 0 \tag{3.3.3}$$

Using the scaling solution $a(\tau) = (-\tau)^{\frac{1}{\epsilon-1}}$, we can express (3.3.3) in the same form as (2.5.7), with the analogue of (2.5.8) in this case taking the form

$$\frac{a''}{a} = \frac{1}{\tau^2}\frac{2-\epsilon}{(1-\epsilon)^2} \equiv \frac{\nu_t^2 - 1/4}{\tau^2}, \qquad \nu_t = \frac{1}{2}\left|\frac{\epsilon-3}{\epsilon-1}\right| \tag{3.3.4}$$

Using the same procedure as that in Section 2.3, we find that the power spectrum of tensor perturbations is

$$\Delta_t^2(k) = \frac{k^3}{2\pi^2}\frac{|f_k|^2}{a^2}\left(\frac{8}{M_{pl}^2}\right) \tag{3.3.5}$$

and the tensor spectral index is

$$n_T = 3 - 2\nu_t = 3 - \left|\frac{\epsilon-3}{\epsilon-1}\right| \tag{3.3.6}$$

Since $\epsilon \gg 1$, we have that $n_T \approx 2$. Therefore, the tensor spectrum is very blue, and so the spectrum of primordial gravitational waves is negligible on present-day scales; moreover, the spectrum is cut off at higher frequencies because (1) gravity is felt in the bulk as well as on the boundary walls and (2) the Universe is kinetic-dominated after ekpyrosis. The 5-dimensional spacetime is locally Minkowski near the brane collision [73], so gravitational waves with wavelengths smaller than the GUT scale leave the horizon after ekpyrosis and are not amplified, since they behave as they would in flat space. This result is in sharp contrast to the prediction from inflation. Consequently, the value of the tensor-to-scalar ratio r provides a significant empirical test for the validity of the cyclic model.

3.4 Alternative Constructions of Bouncing Cosmologies

While we have focused on the ekpyrotic/cyclic model of Turok, Steinhardt *et. al.* in this chapter, this is by no means the only attempt at constructing a bouncing cosmology as a means of circumventing the problems of inflation [29]. We now briefly discuss some of these alternative ideas.

Matter Bounce

The matter bounce picture [84] utilises the duality between the evolution of the canonical fluctuation variables v_k in an inflationary era and those in a contracting phase which is dominated by matter with pressure $P = 0$ (i.e. dust). In inflation, the growing mode of v_k is proportional to $z(\tau)$ and the other mode is decaying. In a contracting phase dominated by matter, the conformal scale factor goes as $a(\tau) \sim \tau^2$ and the solution of the mode equation on super-Hubble scales is

$$v_k(\tau) = c_1\tau^2 + c_2\tau^{-1}, \tag{3.4.1}$$

where c_1, c_2 are constants whose values are determined by initial condition and the second mode is growing, rather than the first. Note that the scaling $v_k(\tau) \sim \tau^{-1}$ arises due to the characteristic equation of state $w = 0$.

Using the conformal Hubble parameter $\mathcal{H} \sim \tau^{-1}$ and the fact that the crossing of the Hubble radius occurs at $k^2 = \mathcal{H}^2$, we have that $\tau_* \sim k^{-1}$. Then the power spectrum of curvature perturbations is

$$\Delta^2_{\mathcal{R}}(k) = \frac{k^3}{z^2(\tau)}|v_k(\tau)|^2 \simeq \frac{k^3}{z^2(\tau)}\left(\frac{\tau_*}{\tau}\right)^2 |v_k(t_i)|^2 \tag{3.4.2}$$

which, after enforcing the vacuum initial conditions, is scale-invariant. However, noting that the actual spectrum should in fact be slightly tilted towards the red, one can add a small cosmological constant Λ which yields the desired result [85]. If thermal initial fluctuations are also considered, then the power spectrum is scale-dependent, but can be made scale-invariant by using a fluid with a particular equation of state ($w = 7/3$) in the contracting phase [86].

Pre-Big Bang Scenario

The pre-Big Bang approach introduces essential elements from superstring theory to produce a cosmological model [69]. In particular, it incorporates (in addition to the graviton) the other massless modes of a closed string, namely the dilaton and the antisymmetric Kalb-Ramond field. To start with, neglecting the latter, the effective action in the string frame is [29]

$$S = \int \mathrm{d}^{d+1}x\,\sqrt{-g}\,e^{-\phi}\left(R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi\right), \tag{3.4.3}$$

where d is the number of spatial dimensions and ϕ is the dilaton. The conformal transformations that take this into thw Einstein frame (where the dilaton is not coupled to the Ricci scalar R) are

$$\tilde{a} = ae^{-\phi(d-1)}, \qquad \tilde{\phi} = \phi\sqrt{\frac{2}{d-1}} \tag{3.4.4}$$

Then the action in the Einstein frame is

$$S = \int \mathrm{d}^{d+1}x \sqrt{-\tilde{g}} \left(\tilde{R} - \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\tilde{\phi}\,\partial_\nu\tilde{\phi} \right) \tag{3.4.5}$$

This action displays a time reflection symmetry ($t \to -t$), and also has a scale factor duality symmetry, which is related to T-duality (see e.g. [49] for details). For a homogeneous, isotropic metric, the symmetry transformation is

$$a \to a^{-1}, \qquad \bar{\phi} \to \bar{\phi}, \tag{3.4.6}$$

where $\bar{\phi} = \phi - 2d\ln a$. Minimally coupling matter (as a perfect fluid) to R and taking $d = 3$, $w = +1/3$, the usual radiation-dominated solution is obtained

$$a(t) \sim t^{1/2}, \qquad \phi(t) = \text{const.} \tag{3.4.7}$$

Taking a time reflection followed by a scale factor duality transformation, a super-exponentially expanding solution is found

$$a(t) \sim (-t)^{-1/2}, \qquad \phi(t) \sim -3\ln(-t), \tag{3.4.8}$$

which corresponds to matter with a 'stringy' equation of state, $w = -1/3$. This corresponds to the pre-Big Bang branch, which represents a contracting phase in the Einstein frame. Note that, in both frames, the constant comoving scales exit the Hubble radius in the pre-Big Bang phase. However, the solution still involves a singularity at $t = 0$, so new (Planck-scale) physics is expected in its vicinity, which reduces to the solutions (3.4.7) and (3.4.8) further out in time.

For other equations of state, it is possible to obtain a pre-Big Bang branch of an expanding solution with reducing curvature. In the setting of pure dilaton gravity, we have [87]

$$a(\tau) \sim \tau^{1/2}, \qquad \tau \to 0^- \tag{3.4.9}$$

Then the mode equation (which is essentially the Mukhanov-Sasaki equation) has super-Hubble solutions

$$v_k(\tau) \sim |\tau|^{1/2}, \qquad v_k(\tau) \sim |\tau|^{1/2}\ln(k|\tau|) \tag{3.4.10}$$

As $\tau \to 0$, the second solution dominates. In contrast to the matter bounce picture, fluctuations are damped on super-Hubble scales so an initial vacuum spectrum is tilted towards the blue rather than the red. The power spectrum is given by

$$\Delta_k^2(v) \sim k^3\,|v_k(\tau_*)|^2\,\frac{\tau}{\tau_*} \tag{3.4.11}$$

$|v_k|^2$ scales as k^{-1} but since $\tau \sim k^{-1}$, $\tau_*^{-1} \sim k$ so $\Delta_k^2(v) \sim k^3$ and $n_s = 4$. Likewise, the tensor modes have a power spectrum with the same gradient. To find a scale-invariant spectrum, one must either deform the background or use the entropic/isocurvature mechanism (as in Section 3.2); in the latter, there exists a natural candidate in superstring theory with which we can identify the second scalar field. This is the axion, which arises as the dual of the antisymmetric Kalb-Ramond field in the Kaluza-Klein compactification of the heterotic string theory to 4 dimensions [88]. Then the desired spectrum is obtained from a scale-invariant spectrum of axion energy density fluctuations (in the presence of dynamical extra dimensions) [89].

3.5 Problems with Bouncing Cosmologies

In this chapter, we have shown how a bouncing cosmology (specifically the ekpyrotic/cyclic model) can be a viable alternative to inflation, and is not encumbered with some of the issues that plague the latter. However, this does not mean that it is completely trouble-free; we now look at some of the challenges facing the model/s we have considered here.

Initial Conditions

While we require initial conditions in order to solve e.g. the mode equation for curvature fluctuations, these initial conditions ideally should not be finely-tuned so as to only work in a very particular environment. While we saw an example of how this is a major problem for inflation, bouncing and cyclic cosmologies specify the initial conditions as far back into the past as possible. We are then confronted with the question of what generated these initial conditions for the background trajectory, which can especially be a problem the cyclic/phoenix Universe picture, where the latter is inherently unstable to quantum fluctuations.

Inhomogeneities

In a contracting Universe, primordial inhomogeneities are expected to be erased through ultra-slow contraction. This is not a problem for the ekpyrotic model; however, one finds that while the contracting phase of a Universe dominated by pressure-free dust can be made sufficiently homogeneous in the primordial state [90], the presence of a cosmological constant (as required in the matter bounce picture for generation of a power spectrum of curvature perturbations which is slightly red-tilted) can cause the perturbations to become unstable [91].

Matching Conditions

As we saw in Section 3.2, curvature perturbations in an ekpyrotic contracting phase need to be matched to those in an expanding phase on the other side of a Big Crunch-Big Bang transition. The particular matching condition (which is related to (3.1.16)) is in fact sensitive to the specific matching surface chosen; any other surface will yield a completely different spectrum in the expanding phase [75, 92]. This corresponds to a form of fine-tuning that is undesirable in a theoretically solid model of cosmology, and we will see in Chapter 4 how this is resolved with a mechanism that does not require such fine-tuning.

Moreover, the models explored in this chapter still involve a singularity at $t = 0$. It is as yet unclear, from a theoretic perspective, as to what physics would determine the evolution in the vicinity of this region and just how the perturbations could be analytically continued across such a singularity, mild or otherwise. The alternative approach is a nonsingular bounce, which may involve introducing new physics to ensure a consistent description; we will explore this possibility in greater detail in Chapter 4.

Dependence on Fundamental Theory

The ekpyrotic/cyclic model is still dependent on the validity of the higher-dimensional picture from M-theory, which is required to be UV-complete and pathology-free. In particular, one roadblock that remains from this point of view is that the ekpyrotic potential has not been derived from the underlying theory from first principles, and still needs to be inserted by hand. Moreover, a full quantum treatment of the Big Crunch-Big Bang transition has not been made, and so the exact dynamics of the bounce are not completely understood.

Instability of Background Trajectory

While the two-field entropic/isocurvature mechanism does indeed generate a (nearly) scale-invariant spectrum of curvature perturbations, as we showed in Section 3.2, the background trajectory (which goes along $s = 0$ using the notation from that section) does in fact prove to be unstable to fluctuations. This instability was first shown by Tolley and Wesley [76], whose analysis we follow here, in the form presented by Levy, Ijjas and Steinhardt [26].

If we consider the two-field Lagrangian

$$\mathcal{L} = \frac{1}{2}R - \frac{1}{2}G_{ab}(\Phi)g^{\mu\nu}\partial_\mu\Phi^a\,\partial_\nu\Phi^b - V(\Phi), \tag{3.5.1}$$

where G_{ab} is the metric on the scalar field space and $a, b = 1, 2$, then a scaling solution (solution where all contributions to energy density scale with time in the same way, keeping fractional contributions constant) exists if there is a continuous transformation with parameter κ such that

$$\frac{\mathrm{d}\Phi^a}{\mathrm{d}\kappa} = \xi^a(\Phi), \qquad g_{\mu\nu} \to e^\kappa g_{\mu\nu}, \qquad S = \int \mathrm{d}^4x\,\sqrt{-g}\,\mathcal{L} \to e^\kappa S \tag{3.5.2}$$

This symmetry means that we can find a change of variables $(\Phi_1, \Phi_2) \to (\phi, \sigma)$ such that

$$\mathcal{L} = \frac{1}{2}R - \frac{1}{2}(\partial\sigma)^2 - \frac{1}{2}f(\sigma)(\partial\phi)^2 - V_0 e^{-c\phi}h(\sigma), \tag{3.5.3}$$

where we have taken the usual ekpyrotic form for the potential V and the background solution is given by $f(\sigma) = h(\sigma) = 1$, $\sigma = 0$. Introducing the variables

$$(w, x, y, z) \equiv \left(\frac{\sqrt{f(\sigma)}\phi'}{\sqrt{6}\mathcal{H}}, \frac{\sigma'}{\sqrt{6}\mathcal{H}}, -\frac{\sqrt{-V_0 h(\sigma)}e^{-c\phi/2}}{\sqrt{3}\mathcal{H}}, \sigma\right), \tag{3.5.4}$$

where the conformal time τ takes negative values and $\mathcal{H} = a'/a$ is (as before) the conformal Hubble parameter, the equations of motion take the form

$$\begin{aligned} w_{,N} &= 3(w^2 + x^2 - 1)\left(w - \frac{c}{\sqrt{6f(z)}}\right) - \sqrt{\frac{3}{2}}\frac{f_{,z}}{f(z)}xw \\ x_{,N} &= 3(w^2 + x^2 - 1)\left(x + \frac{1}{\sqrt{6}}\frac{h_{,z}}{h(z)}\right) + \sqrt{\frac{3}{2}}\frac{f_{,z}}{f(z)}w^2 \\ z_{,N} &= \sqrt{6}x \end{aligned} \tag{3.5.5}$$

where $N \equiv \ln a$ is the number of e-folds of ekpyrosis, y is eliminated using the Friedmann constraint $w^2 + x^2 - y^2 = 1$ and we recall that the parameter $\epsilon = 1 - \mathcal{H}'/\mathcal{H}^2 = 3(w^2 + x^2)$. Since the Friedmann constraint shows that the fractional contributions of the energies sum to 1, we see that ϵ is the sum of the kinetic energies of ϕ and σ (up to a factor of 3). The fixed-point scaling solution for the system of equations (3.5.5) is

$$(w, x, y, z) = \left(\frac{c}{\sqrt{6}}, 0, \sqrt{\frac{c^2}{6} - 1}, 0\right), \tag{3.5.6}$$

where $\sigma = 0$ and only ϕ is dynamical, subject to the constraint $h_{,\sigma}(0) = \frac{-c^2 f_{,\sigma}(0)}{c^2-6}$. As shown by Tolley and Wesley, the perturbation spectrum depends on the remaining degrees of freedom $\{c; f_{,\sigma}(0); f_{,\sigma\sigma}(0); h_{,\sigma\sigma}(0)\}$, since the background solution has $\sigma = 0$. We now take the explicit example presented by them (and considered by Levy, Ijjas and Steinhardt), with

$$\mathcal{L} = \frac{1}{2}R - \frac{1}{2}(\partial\Phi_1)^2 - \frac{1}{2}(\partial\Phi_2)^2 - \tilde{V}_0 e^{-c_1\Phi_1} - \tilde{V}_0 e^{-c_2\Phi_2} \tag{3.5.7}$$

having the scaling solution $\Phi_i = A_i \ln|\tau| + B_i$ and $c_1 A_1 = c_2 A_2$. In this case, the metric on the scalar field space is simply Euclidean, $G_{ab} = \delta_{ab}$. Here the change of variables taking (3.5.7) to the form of (3.5.3) is

$$\phi = \frac{c_2\Phi_1 + c_1\Phi_2}{\sqrt{c_1^2 + c_2^2}}, \qquad \sigma = \frac{c_1\Phi_1 - c_2\Phi_2}{\sqrt{c_1^2 + c_2^2}} + \sigma_0, \tag{3.5.8}$$

where $\sigma_0 = \frac{2\ln(c_2/c_1)}{c_1^2+c_2^2}$. The functions f and h take the form

$$f(\sigma) = 1, \qquad h(\sigma) = 1 + \frac{c^2}{2}\sigma^2 + \mathcal{O}(\sigma^3) \tag{3.5.9}$$

and

$$\frac{1}{c^2} = \frac{1}{c_1^2} + \frac{1}{c_2^2}, \qquad V_0 = \left(\left(\frac{c_2}{c_1}\right)^{\frac{2c_1^2}{c_1^2+c_2^2}} + \left(\frac{c_1}{c_2}\right)^{\frac{2c_2^2}{c_1^2+c_2^2}}\right) \tag{3.5.10}$$

Linearising the background (3.5.5) about the fixed point (3.5.6), the perturbations satisfy

$$\begin{pmatrix} \delta w_{,N} \\ \delta x_{,N} \\ \delta z_{,N} \end{pmatrix} = M \cdot \begin{pmatrix} \delta w \\ \delta x \\ \delta z \end{pmatrix}, \tag{3.5.11}$$

where $(\delta w, \delta x, \delta z) \equiv (w - c/\sqrt{6}, x, z)$ and M is given by

$$M \equiv \begin{pmatrix} \frac{1}{2}(c^2-6) & 0 & 0 \\ 0 & \frac{1}{2}(c^2-6) & \frac{c^2}{2\sqrt{6}}(c^2-6) \\ 0 & \sqrt{6} & 0 \end{pmatrix} \tag{3.5.12}$$

Now, the solution would be stable if the eigenvalues of M were all positive. In this case, the eigenvalues of M are

$$\lambda = \frac{c^2-6}{2}, \qquad \frac{(c^2-6) \pm \sqrt{9c^4 - 60c^2 + 36}}{4} \tag{3.5.13}$$

The baseline ekpyrotic case $\epsilon > 3$ requires $c > \sqrt{6}$, in which case the final eigenvalue is negative (although, as mentioned earlier, ekpyrotic contraction is characterized by $\epsilon \gg 1$, which only makes this conclusion firmer). Hence, the system must be finely tuned to be at the fixed point, otherwise the perturbations carry its trajectory further away, implying an unstable background solution. Moreover, this fine-tuning would have to occur during each cycle of cosmic evolution, which represents a serious problem for the two-field cyclic Universe model. Although we have not discussed non-Gaussianity, we note that this solution also predicts a level of non-Gaussianity [93], $f_{NL} \geq \mathcal{O}(10)$, that is in tension with the bounds from observational data [94]. We will see how the new iteration of a cyclic Universe resolves these problems in Chapter 4.

3.6 Discussion

In this chapter, we presented bouncing cosmology as a plausible alternative to the inflationary paradigm, with a focus on the ekpyrotic/cyclic/phoenix model of Turok, Steinhardt, Lehners *et. al.* We started by showing how this can solve the horizon, flatness and entropy problems, while making the singularity at $t = 0$ milder (as a momentary collapse and rebound of only the orbifold dimension in the higher-dimensional setting). We then proceeded to demonstrate how, using the entropic/isocurvature mechanism with two scalar fields, a (nearly) scale-invariant spectrum of entropy perturbations could be obtained and subsequently converted into a scale-invariant spectrum of curvature perturbations; we also illustrated the means by which the spectrum of tensor perturbations predicted negligible primordial gravitational waves, in contrast to the expectation from the inflationary mechanism of Chapter 2. We also briefly presented the ideas behind the matter bounce and pre-Big Bang scenarios, as alternative constructions of a bouncing cosmology.

We then considered some of the problems faced by these models, with two issues standing out in particular. The first was the dependence of the ekpyrotic mechanism on a higher-dimensional (M-theory) setting, and the absence of a derivation of the potential from first principles. The actual mechanism governing the bounce is also yet to be given a full quantum treatment and completely understood. The second issue was the instability of the background trajectory to fluctuations, given the two-field mechanism that was used to generate the scale-invariant spectrum of curvature perturbations. We use these two problems as motivation for presenting the new, classical form of a cyclic Universe proposed by Ijjas and Steinhardt, which we will explore in Chapter 4.

The Cyclic Universe – A New Perspective 4

In Chapter 3, we demonstrated how the problems of the hot Big Bang model could be solved by taking a bouncing cosmology, with particular focus on the cyclic/phoenix Universe of Turok, Steinhardt *et. al.* The horizon and flatness problems were shown to be resolved through a period of ultra-slow (ekpyrotic) contraction preceding a bounce to an expanding phase. The problem of large-scale structure formation was resolved through the two-scalar field entropic/isocurvature mechanism, whereby a nearly scale-invariant spectrum of entropy perturbations, produced in the contracting phase, was converted into a nearly-scale invariant spectrum of curvature fluctuations in the kinetic-dominated phase after contraction and prior to the bounce. However, this model exhibited two major problems – the background trajectory in the contracting phase was shown to be unstable to fluctuations, and the description of the bounce was still singular (and hence dependent on the fundamental M-theory picture we discussed).

In this Chapter, we will present a rather different view of the cyclic Universe. Although the contracting phase incorporates much of the dynamics proposed by the older model (i.e., a scalar in a steep negative potential), the mechanism for generating entropic perturbations (leading to curvature ones) will be stable for generic initial conditions. We note that this is not necessarily unique to the new picture, and could be implemented in the context of the older model as well. Moreover, the bounce will be non-singular and will occur entirely in the classical regime, in contrast to the string-inspired picture described in Chapter 3.

4.1 Resolving the Background Instability Problem

As we described in Section 3.5, Tolley and Wesley [76] showed that, while the two-scalar field isocurvature model would generate a scale-invariant spectrum of entropic perturbations in a contracting Universe, the background trajectory would be inherently unstable to fluctuations, and so preventing this instability would require a great level of fine-tuning during each cycle.

However, Levy, Ijjas and Steinhardt [26] presented a counter-example by expanding on the action first discussed by Li [95]. We will now demonstrate how their argument yields a stable trajectory without fine-tuning the initial conditions – note that this is not unique to the new model, and indeed could be applied to the old one. Consider the Lagrangian

$$\mathcal{L} = \frac{1}{2}R - \frac{1}{2}(\partial\psi)^2 - \frac{1}{2}e^{-\lambda\psi}(\partial\chi)^2 - \tilde{V}_0 e^{-\lambda\psi}, \tag{4.1.1}$$

where $\lambda > 0$, $\tilde{V}_0 < 0$. The exponential coupling factor to the kinetic term for χ indicates that the the scalar field space metric is no longer Euclidean (as in the decoupled case discussed in Chapter 3). Note that, for now, the field χ is taken to be massless (we will return to this point later, when considering the robustness of ekpyrotic contraction, given the modified Lagrangian). Let us first consider the case $V(\psi) = 0$, and look at background trajectories for different initial conditions in the space parameterised by

$$(\tilde{w}, \tilde{x}, \tilde{y}, \tilde{z}) = \left(\frac{\psi'}{\sqrt{6}\mathcal{H}}, \frac{e^{-\lambda\psi/2}\chi'}{\sqrt{6}\mathcal{H}}, \frac{-a\sqrt{-\tilde{V}_0}e^{-\lambda\psi/2}}{\sqrt{3}\mathcal{H}}, e^{-\lambda\psi/2}(\chi - \chi_0)\right), \qquad \tilde{y} \to 0 \tag{4.1.2}$$

Recall that primes denote derivatives with respect to the conformal time τ, and $\mathcal{H} = a'/a$ is the conformal Hubble parameter. The superscript $\sim$ refers to quantities in terms of (ψ, χ), as opposed to those in terms of (ϕ, σ) (c.f. (3.5.8)), which are obtained by a change of variables. The fixed point solutions $(\tilde{w}, \tilde{x}, \tilde{z}) = (\pm 1, 0, 0)$ are those wherein, if the background solution starts there, it stays there (hence the term 'fixed point'). For generic initial conditions which do not correspond to a fixed point, we refer to Fig. 1 of [26], which shows that the background solution appears to be carried towards a strong attractor curve with the significant property that $\tilde{x} = 0 \to \chi' = 0$, which implies that the kinetic energy of the χ field is negligible. From [76], this corresponds to the state when the adiabatic and entropic perturbations decouple. Such a curve is referred to as a 'fixed curve attractor', and the solutions along this curve are scaling since $\tilde{w}, \tilde{x}$ and $\tilde{y}$ are constant. The negative eigenvalue that arises in the linearised equations of motion points to the existence of a fixed curve attractor, rather than a fixed point. The curve then implies that stability holds for generic initial conditions, and that fine-tuning is no longer demanded of the model. In analogy to (3.5.8), the change of variables for $(\psi, \chi) \to (\phi, \sigma)$ is

$$\psi = \phi + \frac{2}{\lambda}\ln\left[\operatorname{sech}\left(\frac{\lambda\sigma}{2}\right)\right] + \psi_0, \qquad \chi = \frac{2}{\lambda}e^{(\phi+\psi_0)/2}\tanh\left(\frac{\lambda\sigma}{2}\right) + \chi_0, \tag{4.1.3}$$

where $\psi_0 \in \mathbb{R}$ is a constant. Substituting these transformations into (4.1.1) yields

$$\mathcal{L} = \frac{1}{2}R - \frac{1}{2}(\partial\sigma)^2 - \frac{1}{2}f(\sigma)(\partial\phi)^2 - V_0 e^{-c\phi}h(\sigma), \tag{4.1.4}$$

with $f(\sigma) = h(\sigma) = \cosh^2\left(\frac{\lambda\sigma}{2}\right)$, $c = \lambda$. Solutions along the fixed-curve attractor still generate a scale-invariant spectrum of entropic perturbations.

Now let us consider $V(\psi) = 0$. Varying the action (4.1.1) with respect to each of ψ and χ yields the field equations

$$\psi'' + 2\mathcal{H}\psi' - \lambda\tilde{V}_0 e^{-\lambda}\psi a^2 + \frac{\lambda}{2}e^{-\lambda\psi}\chi'^2 = 0 \tag{4.1.5}$$

$$\chi'' + 2\mathcal{H}\chi' - \lambda\psi'\chi' = 0 \tag{4.1.6}$$

The Friedmann constraint is

$$\mathcal{H}^2 = \frac{1}{6}\left(\psi'^2 + e^{-\lambda\psi}\chi'^2 + 2a^2\tilde{V}_0 e^{-\lambda\psi}\right) \tag{4.1.7}$$

Using the variables $(\tilde{w}, \tilde{x}, \tilde{y}, \tilde{z})$, these equations become

$$\tilde{w}_{,N} = 3\left(\tilde{w}^2 + \tilde{x}^2 - 1\right)\left(\tilde{w} - \frac{\lambda}{\sqrt{6}}\right) - \sqrt{\frac{3}{2}}\lambda\tilde{x}^2 \tag{4.1.8}$$

$$\tilde{x}_{,N} = 3\tilde{x}\left(\tilde{w}^2 + \tilde{x}^2 - 1\right) + \sqrt{\frac{3}{2}}\lambda\tilde{w}\tilde{x} \tag{4.1.9}$$

$$\tilde{z}_{,N} = -\sqrt{\frac{3}{2}}\lambda\tilde{w}\tilde{z} + \sqrt{6}\tilde{x} \tag{4.1.10}$$

If $V \neq 0$, then there are 3 fixed point solutions

$$(\tilde{w}, \tilde{x}, \tilde{y}, \tilde{z}) = (-1, 0, 0, 0), \quad (+1, 0, 0, 0), \quad \left(\frac{\lambda}{\sqrt{6}}, 0, \sqrt{\frac{\lambda^2}{6} - 1}, 0\right), \tag{4.1.11}$$

all of which are associated with negative eigenvalues, and so are unstable. The third solution bisects two fixed-curve solutions

$$(\tilde{w}, \tilde{x}, \tilde{y}, \tilde{z}) = \left(\frac{\lambda}{\sqrt{6}}, 0, \sqrt{\frac{\lambda^2}{6} - 1}, \pm\tilde{Z}\right), \tag{4.1.12}$$

where $\tilde{Z} \propto e^{-\lambda^2 N/2}$. Both fixed curves generate a scale-invariant spectrum of entropic perturbations. The previous case $V = \tilde{y} = 0$ corresponds to the particular solution with $\lambda = \pm\sqrt{6}$.

In terms of (ϕ, σ), as before, (4.1.1) becomes (4.1.4), with $f(\sigma) = h(\sigma) = \cosh^2\left(\frac{\lambda\sigma}{2}\right)$, $c = \lambda$, $V_0 = \tilde{V}_0 e^{-\lambda\psi_0}$. The equations of motion then take the form

$$w_{,N} = 3(w^2 + x^2 - 1)\left(w - \frac{\lambda}{\sqrt{6}}\operatorname{sech}(\lambda z)\right) - \sqrt{6}c\tanh(\lambda z)xw \tag{4.1.13}$$

$$x_{,N} = 3(w^2 + x^2 - 1)\left(x + \sqrt{\frac{2}{3}}\lambda\tanh(\lambda z)\right) + \sqrt{6}c\tanh(\lambda z)w^2 \tag{4.1.14}$$

$$z_{,N} = \sqrt{6}x \tag{4.1.15}$$

Given the change of variables $(\psi, \chi) \to (\phi, \sigma)$, the implied transformation for $(w, x, y, z) \to (\tilde{w}, \tilde{x}, \tilde{y}, \tilde{z})$ is

$$\begin{aligned} \tilde{w} &= \operatorname{sech}\left(\frac{\lambda z}{2}\right)w - \tanh\left(\frac{\lambda z}{2}\right)x \\ \tilde{x} &= \tanh\left(\frac{\lambda z}{2}\right)w + \operatorname{sech}\left(\frac{\lambda z}{2}\right)x \\ \tilde{y} &= y, \qquad \tilde{z} = \frac{2}{\lambda}\sinh\left(\frac{\lambda z}{2}\right) \end{aligned} \tag{4.1.16}$$

In terms of (w, x, y, z), the fixed point solutions are

$$(-1, 0, 0, 0), \qquad (+1, 0, 0, 0), \qquad \left(\frac{\lambda}{\sqrt{6}}, 0, \sqrt{\frac{\lambda^2}{6} - 1}, 0\right) \tag{4.1.17}$$

and the fixed curve solutions are

$$\left(\frac{\lambda}{\sqrt{6}}\operatorname{sech}\left(\frac{\lambda z}{2}\right), -\frac{\lambda}{\sqrt{6}}\tanh\left(\frac{\lambda z}{2}\right), \sqrt{\frac{\lambda^2}{6} - 1}, \pm Z\right), \tag{4.1.18}$$

where $Z = \left(\frac{2}{\lambda}\right)\sinh^{-1}\left(\frac{\lambda\tilde{Z}}{2}\right)$. The fixed curves lie on the surface of the cylinder $w^2 + x^2 = \frac{\lambda^2}{6}$. To explicitly show the existence of a negative eigenvalue, one can linearise the equations of motion (4.1.8)–(4.1.10) about the fixed point bisecting the fixed curves and take $(\delta\tilde{w}, \delta\tilde{x}, \delta\tilde{z}) = \left(\tilde{w} - \frac{\lambda}{\sqrt{6}}, \tilde{x}, \tilde{z}\right)$ so that

$$\tilde{M} \equiv \begin{pmatrix} \frac{\lambda^2}{2} - 3 & 0 & 0 \\ 0 & \lambda^2 - 3 & 0 \\ 0 & \sqrt{6} & -\frac{\lambda^2}{2} \end{pmatrix} \tag{4.1.19}$$

Following the same procedure with w, x, z and $(\delta w, \delta x, \delta z) = \left(w - \frac{\lambda}{\sqrt{6}}, x, z\right)$ yields

$$M \equiv \begin{pmatrix} \frac{\lambda^2}{2} - 3 & 0 & 0 \\ 0 & \frac{\lambda^2}{2} - 3 & \frac{\lambda^2(\lambda^2-3)}{2\sqrt{8}} \\ 0 & \sqrt{6} & 0 \end{pmatrix} \tag{4.1.20}$$

Both matrices have the eigenvalues $\left\{-\frac{\lambda^2}{2}, \frac{\lambda^2}{2} - 3, \lambda^2 - 3\right\}$, in which case it is apparent that the first eigenvalue will always be negative. In terms of $(\tilde{w}, \tilde{x}, \tilde{y}, \tilde{z}$, the corresponding eigenvector is parallel to the $\tilde{z}$ unit vector, which is tangent to the appropriate fixed curve (4.1.12). In terms of (w, x, y, z), the eigenvector is parallel to $\hat{z} - \frac{\lambda^2}{2\sqrt{6}}\hat{x}$ (where $\hat{x}, \hat{z}$ are unit vectors) which is tangent to the fixed curve (4.1.18) as well.

At this point, we note that we can generalise this approach to actions without shift symmetry as well

$$\mathcal{L} = \frac{1}{2}R - \frac{1}{2}(\partial\psi)^2 - \frac{1}{2}e^{-\lambda\psi}(\partial\chi)^2 - (1 + r(\chi))V_0 e^{-\mu\psi} + q(\chi) \tag{4.1.21}$$

In particular, the addition of $r(\chi)$ and $q(\chi)$ breaks the shift symmetry because

$$\begin{aligned} V &= (1 + r(\chi))V_0 e^{-\mu\psi} + q(\chi) \\ &\to \left(1 + r\left(e^{\lambda\kappa/2\mu}\chi\right)\right) V_0 e^{-\mu\psi-\kappa} + r\left(e^{\lambda\kappa/2\mu}\chi\right) \\ &\neq e^{-\kappa}V \end{aligned}$$

The most general ekpyrotic, two-field Lagrangian (with shift symmetry, which implies the existence of scaling solutions) which is either a fixed-point or a fixed-curve attractor, and generates a scale-invariant spectrum of entropy perturbations is in the form of (4.1.4), with $V_0 < 0$, $c \in \mathbb{R}$, $h(\sigma) > 0$, $f(\sigma) > 0$. The requirements of shift symmetry, scaling solution, fixed curve attractor and scale-invariant perturbations imply the properties

1. $\lim_{|\sigma|\to\infty} f(\sigma) \to \infty$ monotonically
2. $\lim_{|\sigma|\to\infty} f(\sigma) \propto e^{-\mu\sigma}$
3. $\lim_{|\sigma|\to\infty} (w, x, y, z) = \left(0, \frac{\mu}{\sqrt{6}}, \sqrt{\frac{\mu^2}{6} - 1}, -\text{sign}(\mu)\infty\right) \to$ scaling solution
4. $|\mu| > \sqrt{6}$ for ekpyrosis

Then the remaining degrees of freedom are c, V_0 and $f(\sigma)$ at late times (given that property 1 is satisfied). We will not go through the explicit calculation fully here, but it is found that, perturbing the Einstein equations about the fixed curve solution shown by property 3 in the longitudinal gauge, 3 cases are obtained – fast growth ($|f_{,\sigma}/f| \gtrsim 1$), slow growth ($|f_{,\sigma}/f| \lesssim 1$) and 'just-so growth' ($|f_{,\sigma}/f| \sim 1$) (see [26] for details). Note that the relational operators have the following meanings

$$|A(\tau)| \gtrsim |B(\tau)| \quad \text{if} \quad \frac{\mathrm{d}\ln|A(\tau)|}{\mathrm{d}\ln(-\tau)} < \frac{\mathrm{d}\ln|B(\tau)|}{\mathrm{d}\ln(-\tau)} \quad \text{as}\, \tau \to 0$$
$$|A(\tau)| \sim |B(\tau)| \quad \text{if} \quad \frac{\mathrm{d}\ln|A(\tau)|}{\mathrm{d}\ln(-\tau)} = \frac{\mathrm{d}\ln|B(\tau)|}{\mathrm{d}\ln(-\tau)} \quad \text{as}\, \tau \to 0$$

From [26], fast and slow growth are unable to yield scale invariance. 'Just-so' growth implies that $f(\sigma) = e^{-\lambda\sigma}$, $\lambda \in \mathbb{R}$ such that $\text{sign}(\lambda) = \text{sign}(\mu)$. Then

$$n_s - 1 = 3 - \left|2\left(\frac{\lambda\mu - 2}{\mu^2 - 2}\right) + 1\right| \tag{4.1.22}$$

so scale invariance ($n_s - 1 = 2$) is achieved for $\lambda = \mu$. Note that, in this case,

$$\mathcal{L} = \frac{1}{2}R - \frac{1}{2}(\partial\psi)^2 - \frac{1}{2}e^{-\lambda\psi}(\partial\chi)^2 - V_0 e^{-c\chi}e^{-\mu\psi} \tag{4.1.23}$$

which is equivalent to (4.1.21) with $r(\chi) = e^{-\lambda\chi} - 1$, $q(\chi) = 0$, $c = \lambda$. Although we have not discussed non-Gaussianity in any detail in this dissertation, we note that this action is also predicted [25] to produce small non-Gaussianity ($f_{NL} \sim 5$), which is within the constraints placed by observational data [94].

Given the new exponential coupling in the kinetic term of χ that differentiates it from the old model, we would not be remiss to check how this affects the robustness of ekpyrotic contraction. We can, in general, add a potential term $U(\chi)$ to this effect. It was shown in [96], using non-perturbative numerical simulations, that if χ is massless, then the evolution is rapidly deflected away from the FLRW stationary point if the characteristic mass scale for $V(\psi)$, $M = 1/\lambda$ is too close to the Planck scale, or if the 'fast-roll' parameter ϵ_ψ (which is directly related to its equation of state) is not large enough. Then the evolution of the system tends towards a Kasner-like solution where the spatial gradient of χ, $\bar{S}^x_\chi$, is non-zero and time-independent. This in turn would have a detrimental impact on the homogeneity and isotropy of the background.

For a generic model with smaller values of the characteristic mass scale, $M \lesssim 0.1$, or weakly broken shift symmetry in χ (e.g. through a small non-zero mass, yielding a potential

$U(\chi) = \frac{1}{2}m_\chi\chi^2$), the deflection is strongly suppressed such that the evolution tends towards a long-lived state with negligible $\bar{S}^x_\chi$ and the usual FLRW scaling solution. The deflection away from the flat FLRW background solution caused by the non-canonical kinetic coupling occurs on a time scale that is much longer than deemed relevant for cosmological models of interest. Specifically, this time scale is expected to be $\mathcal{O}(100)$ or more e-folds of contraction of the inverse mean curvature Θ, which (by the gauge choice employed in [96]) coincides with the Hubble radius H^{-1} in the homogeneous limit. However, a non-singular bounce which connects the Universe to a hot expanding phase is expected to take place well before that time.

The final point that we should check in the context of the non-canonical kinetic coupling is the conversion of a scale-invariant spectrum of entropy fluctuations to that of curvature fluctuations. It was proposed in [97] that this would require the inclusion of a quartic kinetic term which is also exponentially coupled to ψ,

$$\mathcal{L} = \frac{1}{2}R - \frac{1}{2}(\partial\psi)^2 - \frac{1}{2}\Sigma_1(\psi)(\partial\chi)^2 + \frac{1}{4}\Sigma_2(\psi)(\partial\chi)^4 - V(\psi,\chi), \tag{4.1.24}$$

with $\Sigma_1(\psi) = e^{-\lambda\psi} = e^{-\psi/m}$, $\Sigma_2(\psi) = e^{\alpha\psi}$, $V(\psi,\chi) = -V_0 e^{-\psi/M} + \frac{1}{2}m_\chi\chi^2$. As the ekpyrotic contracting phase comes to an end and the parameter $\epsilon \to 3$, the kinetic energies of the scalar fields come to dominate the evolution, with the quartic term playing an especially significant role. In [98], it was shown that perturbations in χ sourced entropy modes as

$$\mathcal{S} \equiv H\left(\frac{\delta P}{\dot{P}} - \frac{\delta\rho}{\dot{\rho}}\right) \equiv H\frac{\delta P_{\rm nad}}{P}, \tag{4.1.25}$$

with the non-adiabatic pressure contribution on hypersurfaces of constant density defined as $\delta P_{\rm nad} \equiv \delta P - \frac{\dot{P}}{\dot{\rho}}\delta\rho \neq 0$. Then, on large scales (where $k \ll a|H|$), stress-energy conservation yields the following relation for the evolution of the curvature perturbations

$$\dot{\mathcal{R}} \approx -3H\frac{\dot{P}}{\dot{\rho}}\mathcal{S} = H\frac{\delta P_{\rm nad}}{\rho + P} \tag{4.1.26}$$

Then, in the spatially flat gauge, we obtain the following expression for the curvature perturbation in terms of the scalar perturbations

$$\mathcal{R} \equiv H\frac{\dot{\psi}\delta\psi + \sqrt{\Sigma_1 + \Sigma_2\dot{\chi}^2}\dot{\chi}\delta\chi}{\dot{\psi}^2 + \left(\Sigma_1 + \Sigma_2\dot{\chi}^2\right)\dot{\chi}^2} \tag{4.1.27}$$

with the non-adiabatic pressure being

$$\delta P_{\rm nad} \approx 2c_s^2\left[-\Sigma_2\dot{\chi}^3\dot{\psi}\dot{\mathcal{F}} + \left(\left(\Sigma_1 + \Sigma_2\dot{\chi}^2\right)\left(V_{,\psi} + \frac{1}{4}\Sigma_{2,\psi}\dot{\chi}^4\right)\dot{\chi} - V_{,\chi}\dot{\psi} - \Sigma_2\dot{\chi}^3\ddot{\psi}\right)\mathcal{F}\right] \tag{4.1.28}$$

The relative field fluctuations are quantified by

$$\mathcal{F} \equiv \left(\frac{\dot{\psi}\dot{\chi}}{\dot{\psi}^2 + \Sigma_1\dot{\chi}^2 + \Sigma_2\dot{\chi}^4}\right)\left(\frac{\delta\chi}{\dot{\chi}} - \frac{\delta\psi}{\dot{\psi}}\right) \tag{4.1.29}$$

and the speed of the adiabatic fluctuations is

$$c_s^2 \equiv \frac{\dot{\psi}^2 + \Sigma_1\dot{\chi}^2 + \Sigma_2\dot{\chi}^4}{\dot{\psi}^2 + \Sigma_1\dot{\chi}^2 + 3\Sigma_2\dot{\chi}^4} \tag{4.1.30}$$

Since $\dot{\chi} \to 0$ in the contracting phase, $\delta P_{\text{nad}} \approx 0$. However, in this transitory kinetic-dominated exit phase, $\dot{\chi} \neq 0$, and so $\mathcal{F}$ yields $\delta P_{\text{nad}} \neq 0$, which sources the curvature fluctuations on super-Hubble scales. The spectrum is (nearly) scale-invariant since the sourcing entropy spectrum is itself (nearly) scale-invariant. For the purpose of comparison against previous models, the amplitude of the power spectrum is given by

$$\langle\mathcal{R}^2(x)\rangle = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\left(\frac{\mathcal{R}}{k^\nu}\right)^2 = \int \frac{\mathrm{d}k}{k}\left(\frac{\mathcal{R}^2}{2\pi^2}\right)k^{3-2\nu} = \int \frac{\mathrm{d}k}{k}\,\Delta^2_{\mathcal{R}}(k), \tag{4.1.31}$$

where ν takes its meaning from the Mukhanov-Sasaki equation and its solution in terms of Hankel functions, as before,

$$\begin{aligned} \frac{\nu^2 - 1/4}{\tau^2} &= \frac{z''}{z} = \frac{1}{4}\left(1 + 2\frac{\frac{M}{m}\epsilon - 1}{\epsilon - 1}\right)^2 \\ n_s - 1 = 3 - 2\nu &= 2\left(1 - \frac{\frac{M}{m}\epsilon - 1}{\epsilon - 1}\right) \end{aligned} \tag{4.1.32}$$

Therefore, the quartic kinetic term plays a significant role in the conversion of entropic to curvature fluctuations, although it also seems to serve a second, key purpose. From the point of view of ψ, this is a potential term which is exponential in nature, and describes the sharp increase of the potential towards zero after the contracting phase ends. In the older cyclic models covered in Chapter 3, this was achieved by the function $F(\phi)$ in (3.1.15), which was proposed to be a super-exponential function having its origin in a string-based mechanism (for example, in [22, 23], it was proposed that $F \propto e^{-1/g_s}$, where the string coupling $g_s \propto e^{\gamma\phi}$ for $\gamma > 0$). On the other hand (as we shall see in the next section), this model is supposed to have no string motivation or influence, and so the previous explanation cannot be used – in fact it might be inferred that the form of the potential required for a cyclic Universe becomes less physically motivated in this new context. We can now see that the role of bringing the potential to zero can be played by this effective potential term, although an open question remains as to how we could 'stitch' the two components of the potential together using some well-founded physical motivation.

4.2 Rewriting the Rules of the Game – A Non-Singular Bounce?

Up to this point, all the models we have considered (inflation, cyclic, phoenix) involved a singularity at $t = 0$, in the vicinity of which a quantum description of gravity was expected to take over. In fact, we will now show that this is not necessarily the only way to describe a transition from a contracting to an expanding Universe. Recall the first and second (or Raychaudhuri) Friedmann equations for a spatially flat FLRW universe, (2.1.2)–(2.1.3). For a perfect fluid, we have that the stress-energy tensor takes the form $T_{\mu\nu} = (\rho + P)u_\mu u_\nu + P g_{\mu\nu}$. The null energy condition (NEC) states that $T_{\mu\nu} l^\mu l^\nu \geq 0$ for a null vector l^μ. Inserting the above form of $T_{\mu\nu}$ into the left-hand side of the NEC expression yields[1]

$$T_{\mu\nu} l^\mu l^\nu = (\rho + P)u_\mu u_\nu l^\mu l^\nu + P g_{\mu\nu} l^\mu l^\nu = (\rho + P)u_\mu u_\nu l^\mu l^\nu = (\rho + P)\left(u_\mu l^\mu\right)^2 \geq 0$$
$$\Rightarrow \rho + P \geq 0$$

Now inserting (2.1.2) and (2.1.3), we find that

$$\rho + P = 3\left(\frac{\dot{a}^2}{a^2}\right) - \left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) = 2\left(\frac{\dot{a}^2}{a^2} - \frac{\ddot{a}}{a}\right) = -2\dot{H} \geq 0$$

Hence, the NEC implies that $\dot{H} \leq 0$. If we consider a phase of ordinary (NEC-obeying) contraction, H becomes more negative as the scale factor $a(t)$ reduces and the energy density $\rho \propto H^2$ increases. Conversely, during an expanding phase, H becomes less positive as $a(t)$ increases and $\rho \propto H^2$ reduces. One could then conceive of a situation where these two phases were connected by some classical (non-singular) transition. However, this would entail H increasing from a large negative value to a large positive value (while $\rho \propto H^2$ remains sub-Planckian) at some finite value of $a(t)$. The fact that $\dot{H} > 0$ during the bounce automatically means that the null energy condition must be violated, in order for us to be able to entertain

[1]There is an aside regarding the applicability of the perfect fluid assumption in the model we consider, which we will address shortly.

a classical, non-singular description. While the NEC is a cornerstone of Penrose's singularity theorem [99], it is not a cornerstone of General Relativity (or gravity theory) as such, and violating it can still be considered within the purview of the latter. However, violating the NEC without introducing any pathologies is not straightforwardly achieved – indeed, a scalar field with a canonical kinetic term would not suffice, and nor would a $P(X)$ theory (as it would lead to a classical gradient instability). The model that we will consider is that of a generalised Galileon/Horndeski scalar-tensor theory which yields second-order equations of motion in 4 space-time dimensions and admits an NEC-violating bounce solution[2] [33, 101].

As a brief primer to theories of modified gravity, we introduce Lovelock's theorem, which states that the only possible second-order Euler-Lagrange expression obtainable in 4 space-time dimensions from a scalar Lagrangian density $\mathcal{L} = \mathcal{L}(g_{\mu\nu})$ is [102, 103, 104]

$$E^{\mu\nu} = \alpha\sqrt{-g}\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right) + \beta\sqrt{-g}g^{\mu\nu}, \tag{4.2.1}$$

where α and β are constants,

$$E^{\mu\nu}(\mathcal{L}) = \frac{\mathrm{d}}{\mathrm{d}x^{\rho}}\left(\frac{\partial\mathcal{L}}{\partial g_{\mu\nu,\rho}} - \frac{\mathrm{d}}{\mathrm{d}x^{\lambda}}\left(\frac{\partial\mathcal{L}}{\partial g_{\mu\nu,\rho\lambda}}\right)\right) - \frac{\partial\mathcal{L}}{\partial g_{\mu\nu}} \tag{4.2.2}$$

and $E^{\mu\nu}(\mathcal{L}) = 0$ gives the Euler-Lagrange equations. While the most general action that one can construct purely from the metric is the Eintein-Hilbert action (with a cosmological constant term) plus scalar contractions of the Riemann and Ricci tensors, the only field equations that we obtain in second order or less are the Einstein equations with a cosmological constant. As a result, in order to find a theory of gravity that is different from General Relativity, we must do one or more of –

- Consider fields other than the metric, such as scalars
- Include field equations with higher than second-order derivatives of $g_{\mu\nu}$
- Change the number of space-time dimensions, $D \neq 4$
- Relax the requirements of either rank-2 tensor field equations, symmetry under index exchange, or divergence-free field equations

[2] Generic Horndeski theories introduce non-minimal coupling of the scalar to gravity so that the NEC does not remain well-defined as a condition on matter [100]. Hence, strictly speaking in this context, we talk about violation of the null convergence condition (NCC), $R_{\mu\nu}l^{\mu}l^{\nu} \geq 0$, where $R_{\mu\nu}$ is the Ricci tensor. For minimally-coupled theories, this coincides with the NEC, while its generic violation still corresponds to $\dot{H} > 0$.

- Remove the requirement of locality

As the first of these still preserves the most physical meaning, we will take that option. Specifically, we will work with generalised Galileon/Horndeski theory, which is the most general scalar-tensor theory of gravity in 4 space-time dimensions that yields second-order field equations. The general Horndeski action is given by

$$S = \int \mathrm{d}^4x\,\sqrt{-g}\left(\frac{1}{2}R + \mathcal{L}_H\right), \tag{4.2.3}$$

where $\mathcal{L}_H = \sum_{i=2}^{5}\mathcal{L}_i$ and

$$\begin{aligned} \mathcal{L}_2 = G_2(X,\phi), \qquad \mathcal{L}_3 = G_3(X,\phi)\,\Box\phi \\ \mathcal{L}_4 = G_4(X,\phi)R + G_{4,X}(X,\phi)\left((\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right) \\ \mathcal{L}_5 = G_5(X,\phi)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi - \frac{G_{5,X}}{6}\left((\Box\phi)^3 - 3\,\Box\phi(\nabla_\mu\nabla_\nu\phi)^2 + 2\,(\nabla_\mu\nabla_\nu\phi)^3\right) \end{aligned} \tag{4.2.4}$$

Note that $X = -\frac{1}{2}\nabla^\mu\phi\,\nabla_\mu\phi = -\frac{1}{2}\partial^\mu\phi\,\partial_\mu\phi$ is the canonical kinetic term. We can see that ordinary General Relativity with a single minimally-coupled canonical scalar is recovered by setting $G_2 = X$, $G_3 = G_4 = G_5 = 0$ and the class of $P(X,\phi)$ theories is recovered by taking $G_2 = P(X,\phi)$, $G_3 = G_4 = G_5 = 0$. While we assumed that the stress-energy tensor took the form of that for a perfect fluid in our derivation of the implication $\dot{H} \leq 0$ from NEC violation, $T_{\mu\nu}$ in fact does not correspond to a perfect fluid for a generic Horndeski theory. However, the homogeneous part does take the required form due to the spherical symmetry of the FLRW metric [32], and so we are able make this assumption without trouble. While Ijjas and Steinhardt stress in [33, 101] that there is considerable flexibility in designing non-singular bouncing models within the generalised Galileon framework, we will stick to their example and work with the following coupling functions

$$\begin{aligned} G_2(X,\phi) = k(\phi)X + q(\phi)X^2 - V(\phi), \qquad G_3(X,\phi) = -b(\phi)X \\ G_4(X,\phi) = \frac{1}{2}f_1(\phi) + f_2(\phi)X, \qquad G_5(X,\phi) = 0 \end{aligned} \tag{4.2.5}$$

Before we delve deeper into the dynamics of the non-singular bounce governed by Horndeski theory, let us consider as a brief aside the following – the Lagrangian describing the pre-bounce phases depends on two scalars, with the kinetic term(s) of the second scalar having a non-trivial exponential coupling to the first. On the other hand, the bounce Lagrangian seems to be only dependent on a single (dominant) scalar. Given the generalised first Friedmann equation (3.1.3), one would naïvely expect the bouncing phase to also be dependent on two scalars. This would usually present a sizeable problem as the Horndeski framework has not been concretely generalised to accommodate multiple scalars yet (for a review of approaches to this problem, see [105] and references therein). However, this seemingly troublesome issue is resolved by recognizing that one should consider the relative domination of fields at the level of the background and perturbations. In the scalar field space, one direction sets the trajectory for the background, with the field equations resembling that of the single-field scenario. For the perturbations, one would strictly have to look at both the adiabatic and isocurvature combinations, but as we have shown above, one of the scalars has a much greater mass scale than the other, so that a single-field description is also valid at this level [48].

As usual, we consider the background to be a spatially flat FLRW metric, so that the homogeneous equations of motion that we obtain are [33]

$$\begin{aligned} 3H^2 = &\frac{1}{2}k\dot{\phi}^2 + \frac{1}{4}\left(3q - 2b_{,\phi}\right)\dot{\phi}^4 + V - 3f_{1,\phi}H\dot{\phi} + 3\left(b - 3f_{2,\phi}\right)H\dot{\phi}^3 \\ &- 3\left(f_1 - 3f_2\dot{\phi}^2\right)H^2 \end{aligned} \tag{4.2.6}$$

$$\begin{aligned} -2\dot{H} = &\left(k + f_{1,\phi\phi}\right)\dot{\phi}^2 + \left(q - b_{,\phi} + f_{2,\phi\phi}\right)\dot{\phi}^4 + f_{1,\phi}\ddot{\phi} + \left(3f_{2,\phi} - b\right)\ddot{\phi}\dot{\phi}^2 \\ &- f_{1,\phi}H\dot{\phi} + \left(3b - 11f_{2,\phi}\right)H\dot{\phi}^3 - 4f_2H\dot{\phi}\ddot{\phi} + 6f_2\dot{\phi}^2H^2 \\ &+ 2\dot{H}\left(f_1 - f_2\dot{\phi}\right) \end{aligned} \tag{4.2.7}$$

As before, we have $-\frac{\dot{H}}{H^2} \equiv \epsilon \equiv \frac{3}{2}(1+w)$. Up to linear order, we should make sure that there are no instabilities in the theory. In the ADM formalism,

$$\mathrm{d}s^2 = -N\,\mathrm{d}t^2 + g_{ij}\left(N^i\,\mathrm{d}t + \mathrm{d}x^i\right)\left(N^j\,\mathrm{d}t + \mathrm{d}x^j\right), \tag{4.2.8}$$

where N is the lapse and N^i is the shift. We perturb these as

$$\delta N = N - \bar{N} = \alpha, \qquad \delta N_i = N_i - \bar{N}_i = \partial_i\beta, \tag{4.2.9}$$

where the background metric has $\bar{N} = 1$, $\bar{N}_i = 0$, $\bar{g}_{ij} = a^2\delta_{ij}$. As previously, choosing the unitary gauge where $\delta\phi = 0$, $\mathcal{R}$ is the (gauge-invariant) comoving curvature mode and h_{ij} is the (gauge-invariant) linear tensor perturbation satisfying $\partial_j h_{ij} = 0$, $h^i_i = 0$. Then, to second order, the Lagrangian is

$$\mathcal{L}^{(2)}_{\mathcal{R},\alpha,\beta} = a^3\left(-3A_h\dot{\mathcal{R}}^2 + B_h\frac{(\partial_i\mathcal{R})^2}{a^2} + m_\alpha\alpha^2 + 6\gamma\alpha\dot{\mathcal{R}} - 2A_h\alpha\frac{\nabla^2\mathcal{R}}{a^2} + 2\frac{\nabla^2\beta}{a^2}(A_h\dot{\mathcal{R}} - \gamma\alpha)\right), \tag{4.2.10}$$

where $\nabla^2 = \partial_i \partial^i$ is the Laplacian and

$$A_h(t) = 1 + f_1 - f_2 \dot{\phi}^2, \qquad B_h(t) = 1 + f_1 + f_2 \dot{\phi}^2 \tag{4.2.11}$$

$$m_\alpha(t) = \frac{1}{2} k \dot{\phi}^2 + \frac{1}{2} \left(3q - 2b_{,\phi}\right) \dot{\phi}^4 - 3 f_{1,\phi} \dot{\phi} H + 6 \left(b - 3 f_{2,\phi}\right) H \dot{\phi}^3 - 3 \left(1 + f_1 - 6 f_2 \dot{\phi}^2\right) H^2 \tag{4.2.12}$$

$$\gamma(t) = \left(1 + f_1 - 3 f_2 \dot{\phi}^2\right) H + \frac{1}{2} f_{1,\phi} \dot{\phi} - \frac{1}{2} \left(b - 3 f_{2,\phi}\right) \dot{\phi}^3 \tag{4.2.13}$$

Varying $S^{(2)}$ with respect to N_i yields the momentum constraint (which is also a simple expression for the lapse perturbation),

$$A_h(t) \dot{\mathcal{R}} = \gamma(t) \alpha \tag{4.2.14}$$

By substituting this into the Hamiltonian constraint, we find the equation for the shift perturbation

$$\gamma^2(t) \frac{\nabla^2 \beta}{a^2} = \left(m_\alpha(t) A_h(t) + 3\gamma^2(t)\right) \dot{\mathcal{R}} - \gamma(t) A_h(t) \frac{\nabla^2 \mathcal{R}}{a^2} \tag{4.2.15}$$

If $\gamma = 0$, then either $\dot{\mathcal{R}} = 0$ or $A_h(t) = 0$. By substituting (4.2.14) and (4.2.15) into the second-order Lagrangian (4.2.10) to eliminate the explicit lapse and shift perturbations, we find the second-order action for $\mathcal{R}$,

$$S^{(2)}_{\mathcal{R}} = \int \mathrm{d}^4 x \, a^3 \left(A(t) \dot{\mathcal{R}}^2 - B(t) \frac{(\partial_i \mathcal{R})^2}{a^2} \right), \tag{4.2.16}$$

where

$$A(t) = m_\alpha(t) \left(\frac{A_h(t)}{\gamma(t)} \right)^2 + 3 A_h(t), \qquad B(t) = a^{-1}(t) \frac{\mathrm{d}}{\mathrm{d}t} \left(a(t) \frac{A_h^2(t)}{\gamma(t)} \right) - B_h(t) \tag{4.2.17}$$

Likewise, the second-order action for the tensor modes is

$$S^{(2)}_{h_{ij}} = \int \mathrm{d}^4 x \, a^3 \left(A_h(t) \dot{h}_{ij}^2 - B_h(t) \frac{(\partial_l h_{ij})^2}{a^2} \right) \tag{4.2.18}$$

In the scalar sector, the absence of ghosts requires $A(t) > 0$ and the absence of gradient instabilities requires $B(t) > 0$. Likewise, in the tensor sector, we require $A_h(t)$, $B_h(t) > 0$. This means that the sound speed must obey $c_s^2 = \frac{B(t)}{A(t)} > 0$, while subluminality mandates that $c_s^2 < 1$.

As shown by Ijjas and Steinhardt in [101], we can use this framework to enter an NEC-violating phase, undergo a non-singular bounce and exit without finding instabilities. However, there is some remnant trouble related to the behaviour of $\gamma(t)$ in this region, which was solved in [33] by including the $\mathcal{L}_4$ term in the Horndeski action. Rearranging and integrating the expression for $B(t)$, (4.2.17), from some arbitrary time t to the beginning of the NEC-violating phase at t_0, we have

$$\left.\frac{aA_h^2}{\gamma}\right|_t = \left.\frac{aA_h^2}{\gamma}\right|_{t_0} - \int_t^{t_0} \mathrm{d}t\, a(B + B_h) \equiv \left.\frac{aA_h^2}{\gamma}\right|_{t_0} - I(t) \tag{4.2.19}$$

Without loss of generality, we can take $\gamma(t_0) = \gamma_0 > 0$. If we take $A_h = B_h = 1$ (as in the case of [101], where $\mathcal{L}_4 = 0$), then since $a(t) > 0\,\forall\, t$, we find that as $t \to -\infty$, $I(t)$ diverges and so $\frac{aA_h^2}{\gamma}$ must become negative at some point. If it was initially positive, then the entire fraction passes through 0, so that γ diverges. However, it also implies that γ would pass through 0 at some point, from which it follows that either a or B would diverge. Note here that this unwanted behaviour arises due to the cubic Galileon action and is not connected to the fact that the NEC is being violated; indeed, the pathology occurs before the NEC-violating phase. The significant change, as brought about in [33], is the introduction of a non-trivial quartic interaction $\mathcal{L}_4$, which makes A_h and B_h dynamical and allows (4.2.19) to be satisfied without incurring the wrath of the aforementioned pathology. It is then conceivable that B and B_h approach zero sufficiently quickly, so that the integral $I(t)$ actually converges as $t \to -\infty$. If we take $a(t) \leq |t|^p$ (where $t < t_0$ and p is finite), then the sufficient condition is that B, $B_h \to 0$ faster than $\frac{1}{|t|^{p+1}}$. If γ_0 is chosen such that

$$\left.\frac{aA_h^2}{\gamma}\right|_{t_0} - I(-\infty) > 0,$$

which is possible since $I(-\infty)$ is finite, then the left-hand side of (4.2.19) can stay positive for all $t < t_0$ (which in turn implies that γ can remain positive for all finite t). To ensure that B is finite, the additional requirement is that $\frac{aA_h^2}{\gamma}$ be finite. Given that a diverges as $t \to -\infty$, then $A_h \to 0$ faster than $\frac{1}{|t|^{p/2}}$ is required. If $\gamma \to 0$ as $|t|^q$, then $A_h \to 0$ as $|t|^{q-(p/2)}$ is required. The positivity of A is achieved without affecting B by including a potential term $V(\phi(t)) = V(t)$ that only affects $m_\alpha(t)$ directly.

At this point, a classically non-singular bouncing solution can be explicitly found using the 'inverse method', first proposed in [101]. In the context of the cubic Galileon, this means that the background equations allow us the freedom to specify the behaviour of $H(t)$ and

$\phi(t)$, so that we can infer the behaviour of $A(t)$ and $B(t)$ in particular. While, by using this method, $c_s^2 = \frac{B(t)}{A(t)}$ can be found to be positive during the NEC-violating phase, it turns out that it becomes negative right before the onset of this phase (c.f. Figure 3 in [33]), and this is related to the trouble found with $\gamma(t)$, which was attributed to the cubic Galileon action itself. However, including the $\mathcal{L}_4$ term in the Horndeski action prevents c_s^2 from turning negative at all finite times [33], and there are sufficient degrees of freedom that we can specify $H(t)$, $\gamma(t)$, $A_h(t)$, $B_h(t)$ and $V(t)$, in order to infer the behaviour of $A(t)$ and $B(t)$. The expressions for the coupling functions in terms of the input parameters are

$$f_1(t) = \frac{A_h + B_h}{2} - 1, \qquad f_2(t) = \frac{B_h - A_h}{2\dot{\phi}^2} \tag{4.2.20}$$

$$b(t) = \frac{2}{\dot{\phi}^3}\left(-\gamma + (2A_h - B_h)H + \frac{1}{2}\dot{f}_1 + \frac{3}{2}\dot{f}_2\dot{\phi}^2\right) \tag{4.2.21}$$

$$k(t) = -\frac{2}{\dot{\phi}^2}\left(\dot{\gamma} + 3H\gamma + \frac{\mathrm{d}}{\mathrm{d}t}((A_h + B_h)H) + \frac{3}{2}(A_h + B_h)H^2 + \ddot{f}_1 - 2V\right) \tag{4.2.22}$$

$$q(t) = \frac{4}{3\dot{\phi}^4}\left(\dot{\gamma} - 9H\gamma + \frac{\mathrm{d}}{\mathrm{d}t}((A_h + B_h)H) + \frac{9}{2}(B_h - A_h)H^2 + \ddot{f}_1 - 3V\right) + \frac{2}{3}b_{,\phi} \tag{4.2.23}$$

Given these, we can find m_α as

$$m_\alpha(t) = 3(A_h + B_h)H^2 - 4V + \frac{\mathrm{d}}{\mathrm{d}t}((A_h + B_h)H) + \dot{\gamma} + 3H\gamma + 3H\dot{f}_1 + \ddot{f}_1 \tag{4.2.24}$$

The above relations can be expressed as functions of ϕ using $\phi(t) \to t(\phi)$. Note that f_1, f_2 and b follow from the definitions of A_h, B_h and γ, while k and q are derived using the background equations (4.2.6)–(4.2.7).

Since we would like to have a geodesically complete cosmological model where Horndeski gravity asymptotes to Einstein gravity before and after the bounce, we consider the significance of $\gamma(t)$, which would coincide with $H(t)$ for theories described by a perfect fluid [32]. The presence of a non-trivial $\mathcal{L}_4$ and the non-minimal coupling to gravity means that γ differs from H – this is known in the literature as 'braiding'. The transition between Einstein and Horndeski gravity means that, as mentioned above, γ must pass through zero. It is then found that the lapse perturbation α and the scalar part σ of the shear perturbation tensor $\sigma_{\mu\nu} = \frac{1}{3}K\gamma_{\mu\nu} - K_{\mu\nu}$ (where $\gamma_{\mu\nu}$ is the linearised metric, $K_{\mu\nu}$ is the extrinsic curvature tensor and K is the extrinsic curvature) diverge as $\gamma \to 0$. However, it seems that this problem is only manifest in the unitary and spatially flat gauges. The Newtonian gauge is defined by $\sigma \equiv 0$ and this protects the gauge variables from blowing up (α as well, since it goes as $1/\gamma$, whereas σ goes as $1/\gamma^2$). The linear stability analysis performed in Newtonian gauge reveals that all modes with wavelengths above the Planck scale evolve with $c_s^2 > 0$; for details, see [32].

At this point, let us take a moment to summarise cosmic evolution over the course of a cycle (and indeed multiple cycles) [31]. Starting in the dark energy-dominated era, the positive value of $V(\phi)$ is equal to the observed density of dark energy. The driving scalar ϕ is, as before, assumed to be a quintessence-like field so that accelerated expansion continues until the potential turns negative. Ekpyrotic contraction then begins, and works as a robust smoothing mechanism that solves the horizon and flatness problems; a nearly scale-invariant spectrum of entropic perturbations is formed through the new 2-field isocurvature mechanism described in Section 4.1. As the kinetic energy of the dominant scalar field increases relative to the potential and the 'fast-roll' parameter $\epsilon \to 3$, we exit the contracting phase, during which time the entropic perturbations are converted to curvature perturbations through the mechanism described at the end of Section 4.1. The Universe then undergoes a classical non-singular bounce, where the Hubble parameter goes from a large positive to a large negative value, while the total energy density $\rho \propto H^2$ remains sub-Planckian. The dynamics governing this bounce phase are described by the generalised Galileon/Horndeski Lagrangian presented in this section. After the bounce, radiation (followed by matter) domination takes over, following which we return to dark energy domination, which corresponds once again to accelerated expansion. Note that the Friedmann equation implies that the Hubble radius is given by $r_H = \epsilon_+ \Delta t$, where ϵ_+ is (linearly related to) the equation of state during the expanding phase. If the scale factor $a(t)$ increases by a factor of f during this time, then r_H increases by a factor of $a^{\epsilon_+} = f^2$. Since the bouncing model assumes that the Hubble parameter oscillates between the positive and the negative of the same value, we find that, during the contracting phase, r_H will reduce by the same factor f^2 over a time interval given by $\Delta t' = (\epsilon_+/\epsilon_-)\Delta t$. This implies that the contracting phase lasts for far less time than the expanding phase, so the scale factor reduces by comparatively little. As a result, while a local observer experiences cycles of expansion and contraction, given by the oscillating Hubble parameter, the net result globally is a de Sitter-like evolution as the scale factor essentially increases exponentially over multiple cycles. Energy and matter are massively diluted so that only a minuscule fraction will remain inside the Hubble radius over the following cycle.

4.3 Entropy in the New Cyclic Universe

As we discussed in Section 2.5, the hot Big Bang model presents a sizeable problem from the point of view of entropy distribution and the second law of thermodynamics; this is only exacerbated by the inflationary paradigm. While a natural resolution to this problem was presented in Chapter 3 using the string-based cyclic model, in the form of (3.0.2) implying an upper limit to the collision entropy, we must revisit it in the new framework outlined in this chapter.

The question of entropy was addressed in [106], the argument of which we reconstruct here. After the non-singular bounce occurs, the Universe is radiation-dominated as the scalar's energy density decays, and space-time is homogeneous, isotropic and flat as a result of the preceding ekpyrotic contraction. Hence, the matter-radiation is in thermal equilibrium at nearly maximal entropy, while the gravitational entropy and Weyl curvature are infinitesimally small. Since the expansion from the bounce to the time of recombination is (nearly) adiabatic and homogeneous, the distribution of entropy at recombination remains essentially identical to that after the bounce.

With the continued expansion and cooling of the Universe, the total entropy rises due to nonlinear structure formation through gravitational instability and irreversible processes, i.e. formation of galaxies and other astrophysical systems. A large proportion of this takes the form of Bekenstein-Hawking entropy, which is associated with supermassive black holes that formed after recombination. By the time dark energy is dominant, this contribution exceeds all others overwhelmingly, by a factor of $\sim 10^{15}$. Over the remainder of the dark energy-dominated phase, the entropy densities of radiation and black holes get diluted, but the former more so than the latter. Then, in order to find the entropy contribution from a previous cycle, we only need to follow the evolution of the supermassive black holes. During contraction, since the scale factor reduces by comparatively little, the density of the black holes does not change significantly. As a result, most black holes are sufficiently distanced from a given Hubble patch that undergoes a bounce, and so are outside an observer's Hubble radius a cycle later. If we consider a black hole at a distance L from an observer's worldline in the previous cycle, the condition for that black hole to lie within the Hubble radius at present would be

$$L \times e^{N_{DE}} \times \mathcal{O}(1) \times e^{60} < H_0^{-1}, \tag{4.3.1}$$

where the scale factor expands by 60 e-folds in the radiation- and matter-dominated eras. Then, given the value of the Hubble parameter today, we have $L < 0.004$ cm.

Inevitably, our discussion of the evolution of entropy leads us to think about reconciliation with the second law of thermodynamics, which is done as follows. The entropy within the Hubble radius does decrease exponentially during contraction, but this is only because the Hubble radius is shrinking, so the 'visible' entropy is exiting the horizon in essence. However, the global entropy still increases as the number of black holes and the scale factor globally increase from cycle to cycle (although the entropy and black hole densities remain largely unchanged over the course of multiple cycles). As a result, the second law is not violated from the perspective of this cyclic model.

A question that [106] left unanswered was the evolution of black holes in the near-field regime of a Hubble patch that undergoes a nonsingular bounce. This was investigated in [107], albeit in the context of a different model (a canonical scalar and a massless ghost field, both minimally coupled to gravity). By evolving the system of equations nonlinearly using numerical methods, the authors showed that the local evolution of the bouncing Hubble patch is largely unaffected by the presence of the black hole if its radius is smaller than the minimum Hubble radius by a factor of approximately 3.5. On the other hand, for black holes with initial radius larger than this, the particle horizon and the black hole apparent horizon merge and briefly cease to exist. After the Hubble radius reaches its minimum, the horizons separate again. This does not include the possibility where the minimum size of the Hubble radius is much smaller than the radius of the black hole. As we noted, this investigation was carried out using a different framework to model the NCC-violating period. It would certainly be interesting to understand how the conclusions of [107] would be altered (if at all) when we take into account the Horndeski bounce, especially given the non-minimal coupling to R that is introduced by the presence of $\mathcal{L}_4$. Given the differences in ensuring linear stability that we have found between minimally and non-minimally coupled models, this is certainly not a foregone conclusion that we could reach purely by association. In particular, we expect that some new condition on the stability of the evolution of the black hole would result; whether this turns out to be more or less stringent is a question that remains to be solved.

4.4 Unitarity Violation during the Bounce – An Unresolved Issue?

Up to now, we have considered in some depth the classical description of a period of NEC (or NCC) violation in the framework of generalised Galileon/Horndeski gravity. Specifically, we have shown how the classical instability associated with a negative speed of propagation of the curvature modes (which is a result of the cubic Lagrangian terms) can be avoided by including non-trivial quartic terms in the action. The 'inverse method' can be used to find the expressions of the coupling functions in terms of the input parameters that describe a given non-singular bouncing solution. Moreover, the apparent singularity linked to the behaviour of $\gamma(t)$ can be shown to be a coordinate singularity, which can be removed by working in the Newtonian gauge (rather than the ubiquitous unitary or spatially-flat gauges). However, we have not yet properly considered the requirements enforced by consistency of the quantum-level description. While the background does evolve classically (i.e. at sub-Planckian scales), we must still take note of its energy scale $E_{\rm back}$ in relation to the strong coupling scale of the theory Λ_s. The latter is the level at which we expect perturbative unitarity to be violated in the theory. Then, if this is in fact lower than $E_{\rm back}$, we would be hard-pressed to establish concretely that NEC violation (and hence a non-singular bounce) did actually take place. We will take inspiration from the work of de Rham and Melville in [108] to identify an open question in the new cyclic model that must be addressed fully before we can consider it to be truly robust.

For the purposes of understanding the problem, we use the setup of [108], which is a $P(X,\phi)$ theory in an FLRW background. Perturbing the scalar as $\phi = \bar{\phi} + \varphi$ in an arbitrary gauge, the effective scalar field metric is

$$Z^{\mu\nu}\partial_\mu\varphi\,\partial_\nu\varphi = -A(t)\dot{\varphi}^2 + \frac{B(t)}{a^2}(\partial_i\varphi)^2 \tag{4.4.1}$$

In the WKB approximation, the decomposition of the quantised modes takes the form

$$\hat{\varphi}(t,x^i) = \int \frac{\mathrm{d}^3k_i}{(2\pi)^3a^3}\frac{1}{\sqrt{\mathcal{N}(k)}}\left(\hat{a}^\dagger(k_i)e^{i\left(k\int\frac{c_s(t)}{a(t)}\,\mathrm{d}t-k_ix^i\right)} + \hat{a}(k_i)e^{-i\left(k\int\frac{c_s(t)}{a(t)}\,\mathrm{d}t-k_ix^i\right)}\right), \tag{4.4.2}$$

where, as before, $c_s^2(t) = B(t)/A(t)$. The normalisation $\mathcal{N}(k)$ is found by taking the Klein-Gordon norm along a 3D surface Σ with unit normal vector n^μ and induced metric $\gamma_{\mu\nu}$,

$$|\hat{\varphi}|^2 = -i\int_\Sigma \mathrm{d}^3x\,\sqrt{\gamma}\,n^\mu\left(\hat{\varphi}\overleftrightarrow{\partial_\mu}\hat{\varphi}^\dagger\right) \tag{4.4.3}$$

Taking a spacelike surface $t=$ const., the unit normal vector is $n^\mu = \sqrt{A}\,\delta^\mu_0$ and the induced volume element is $\sqrt{\gamma} = B^{-3/2}$ so that (4.4.3) becomes

$$|\hat{\varphi}|^2 = -i\frac{\sqrt{A}}{B^{3/2}}\int \mathrm{d}^3x\,a^3\left(\hat{\varphi}\overleftrightarrow{\partial_t}\hat{\varphi}^\dagger\right) \tag{4.4.4}$$

Then, given the dispersion relation $F = A\omega^2 - B\frac{k^2}{a^2}$ and $\omega = c_s\frac{k}{a}$, we have

$$\mathcal{N}(k) = \frac{\partial F}{\partial \omega} = 2A\omega = 2(AB)^{1/2}\frac{k}{a} \tag{4.4.5}$$

In the WKB approximation at high energies, we can look at an approximate S-matrix if it is defined over a time scale that obeys $\Delta t \ll 1/E_{\text{back}}$. Specifically, we consider $n \to n$ scattering, with $|i\rangle = |f\rangle = |k_1, \ldots, k_n\rangle$. Recall that unitarity of the S-matrix implies conservation of probabilities, so it is a cornerstone of the quantum description. Taking the usual perturbative expansion $\hat{S} = \hat{\mathbb{1}} + i\hat{T}$, we require

$$\hat{\mathbb{1}} = (\hat{\mathbb{1}} - i\hat{T}^\dagger)(\hat{\mathbb{1}} + i\hat{T}) = \hat{\mathbb{1}} + i\hat{T} - i\hat{T}^\dagger + \hat{T}^\dagger\hat{T}, \tag{4.4.6}$$

which implies that

$$i\hat{T}^\dagger - i\hat{T} = \hat{T}^\dagger\hat{T} \tag{4.4.7}$$

Multiplying this by $\langle i|$ on the left and $|i\rangle$ on the right, the left-hand side of (4.4.7) becomes

$$\langle i|\, i\hat{T}^\dagger - i\hat{T}\,|i\rangle = i\,\langle i|\,\hat{T}\,|i\rangle^* - i\,\langle i|\,\hat{T}\,|i\rangle = 2\,\mathrm{Im}(\langle i|\,\hat{T}\,|i\rangle), \tag{4.4.8}$$

while the right-hand side becomes

$$\langle i|\,\hat{T}^\dagger\hat{T}\,|i\rangle = \sum_{|f\rangle}\langle i|\,\hat{T}^\dagger\,|f\rangle\,\langle f|\,\hat{T}\,|i\rangle = \sum_{|f\rangle}\left|\langle f|\,\hat{T}\,|i\rangle\right|^2 \tag{4.4.9}$$

With $|f\rangle = |q_1, \ldots, q_N\rangle$, we reconcile the result to that obtained in [108],

$$2\,\mathrm{Im}(\langle i|\,\hat{T}\,|i\rangle) = \sum_N \langle i|\,\hat{T}^\dagger\,|q_1, \ldots, q_N\rangle\,\langle q_1, \ldots, q_N|\,\hat{T}\,|i\rangle \geq \left|\langle q_1, \ldots, q_n|\,\hat{T}\,|i\rangle\right|^2 \tag{4.4.10}$$

Now explicitly factoring out the momentum-conserving delta function and taking $\langle f|\,\hat{T}\,|i\rangle = (2\pi)^4\delta^{(4)}(p_f - p_i)\,\langle f|\,\hat{M}\,|i\rangle$, we obtain

$$2\,\mathrm{Im}(\langle i|\,\hat{M}\,|i\rangle)\cdot(2\pi)^4\delta^{(4)}(p_f - p_i) = \sum_{|f\rangle}\left|\langle f|\,\hat{M}\,|i\rangle\right|^2\left((2\pi)^4\delta^{(4)}(p_f - p_i)\right)^2 \tag{4.4.11}$$

Taking a state $|i'\rangle$ which differs from $|i\rangle$ infinitesimally in the net momentum $p_{i'}$, then [109]

$$\langle i'|\,\hat{M}\,|i\rangle = \langle i|\,\hat{M}\,|i\rangle\,, \qquad \langle i'|\,\hat{M}\,|f\rangle \quad \forall\,|f\rangle \tag{4.4.12}$$

Now multiplying $\langle i'|$ and $|i\rangle$ on both sides of (4.4.7), we get

$$\langle i'| i\hat{T}^\dagger - i\hat{T} |i\rangle = (2\pi)^4\delta^{(4)}(p_{i'} - p_i) \langle i| i\hat{M}^\dagger - i\hat{M} |i\rangle = (2\pi)^4\delta^{(4)}(p_{i'} - p_i) \cdot 2\,\mathrm{Im}(\langle i| \hat{M} |i\rangle) \tag{4.4.13}$$

and

$$\begin{aligned} \langle i'| \hat{T}^\dagger\hat{T} |i\rangle &= \sum_{|f\rangle} \langle f| \hat{T} |i'\rangle^* \langle f| \hat{T} |i\rangle \\ &= \sum_{|f\rangle} \left|\langle f| \hat{M} |i\rangle\right|^2 \cdot (2\pi)^4\delta^{(4)}(p_f - p_{i'}) \cdot (2\pi)^4\delta^{(4)}(p_f - p_i) \\ &= (2\pi)^4\delta^{(4)}(p_{i'} - p_i) \cdot \sum_{|f\rangle} \left|\langle f| \hat{M} |i\rangle\right|^2 \cdot (2\pi)^4\delta^{(4)}(p_f - p_i) \end{aligned} \tag{4.4.14}$$

Hence,

$$2\,\mathrm{Im}(\langle i| \hat{M} |i\rangle) = \sum_{|f\rangle} \left|\langle f| \hat{M} |i\rangle\right|^2 \cdot (2\pi)^4\delta^{(4)}(p_f - p_i) \tag{4.4.15}$$

In terms of scattering amplitudes, we explicitly recover the expression from [108],

$$\begin{aligned} 2\,\mathrm{Im}(\mathcal{A}_{2n}(k_j;k_j)) \geq \int \frac{\mathrm{d}^3 q_1}{(2\pi)^3\mathcal{N}(q_1)} \cdots \int \frac{\mathrm{d}^3 q_n}{(2\pi)^3\mathcal{N}(q_n)} \left|\mathcal{A}_{2n}(k_j;q_j)\right|^2 \\ \times \left[(2\pi)^4\delta\left(\frac{c_s^2}{a^2}\sum k_j^2 - \sum q_j^2\right)\delta^{(3)}\left(\sum k_j - \sum q_j\right)\right] \end{aligned} \tag{4.4.16}$$

For the case of a 4-pt function (i.e. $n = 2$), the phase space factors on the right-hand side of (4.4.16) are evaluated in terms of the centre-of-mass energy $\sqrt{s}$ and scattering angle θ, which yields the optical theorem for the 2-2 scattering amplitude $\mathcal{A}_4(s,\theta)$. Using the dispersion relation $\omega^2 = c_s^2 k^2/a^2$ and taking a partial wave expansion in terms of Legendre polynomials, the following bound is implied by the requirement of perturbative unitarity for each $l \geq 0$,

$$|\mathcal{A}_{4,l}(s)| \leq 8\pi^2 \frac{\omega}{k/a} AB = 8\pi^2 (AB^3)^{1/2}, \tag{4.4.17}$$

where

$$\mathcal{A}_{4,l}(s) = \int_{-1}^{1} \mathrm{d}\cos\theta\, P_l(\cos\theta)\mathcal{A}_4(s,\theta)$$

Recall that the setup of [108] was a $P(X,\phi)$ theory. Classically, a bouncing solution would involve $c_s^2 < 0$, which is already seriously problematic; perturbative unitarity would also be unquestionably violated. This was solved by taking $P(X, \phi)$ as an EFT and including higher-order operators which would enter at some cutoff scale Λ_c. This would restore unitarity, albeit with the caveat that Λ_c would have to be low enough to ensure this but high enough to not affect the validity of the EFT. This higher-order operator should be irrelevant and take the form

$$S_{NML} = \int \mathrm{d}t\, \mathrm{d}^3x\, \frac{a^{3-2L}}{\Lambda_{NML}^{N+2M+4L-4}} \varphi^N \dot{\varphi}^M (\partial_i \varphi)^{2L}, \tag{4.4.18}$$

where N, M, L are arbitrary positive integers which satisfy $N + 2M + 4L > 4$. Now consider a cubic operator of the form (4.4.18) with $N + M + 2L = 3$. This is outside the purview of the usual $P(X, \phi)$ theory (so its effect must be considered in the context of an EFT) and is irrelevant, since $N + 2M + 4L > 4$. At fixed θ, this would contribute to the 4-pt function as

$$\mathcal{A}_4^{\text{cubic}} \sim \frac{k^{4L-2}\omega^{2M}}{B\Lambda_{NML}^{2(N+2M+4L-4)}} \sim \frac{A^{2L-1}}{B^{2L}} \left(\frac{\sqrt{s}}{\Lambda_{NML}} \right)^{2(M+2L-1)} \tag{4.4.19}$$

Then using (4.4.17), perturbative unitarity would be violated at

$$\sqrt{s} \sim A^{\frac{3-4L}{4(M+2L-1)}} B^{\frac{3+4L}{4(M+2L-1)}} \Lambda_{NML} \tag{4.4.20}$$

In the case of a $2n$-pt function, the unitarity bound that results is[3]

$$|\mathcal{A}_{2n}| \lesssim A^{(3-n)/2} B^{3(n-1)/2} s^{2-n} \tag{4.4.21}$$

We should note here that the impact of these bounds was evaluated by de Rham and Melville in $P(X, \phi)$ by specifically calculating the scattering amplitudes and considering whether the conditions imposed by unitarity were satisfied for all the coefficients in the decomposition (in terms of Legendre polynomials) of the amplitudes. The problem that this might pose in the context of the cyclic model we have discussed in this chapter was acknowledged by Ijjas, Pretorius and Steinhardt in [100], albeit somewhat indirectly. The approach they used was to ensure that the Einstein-scalar system of partial differential equations (PDEs) satisfied strong hyperbolicity, which they differentiated from weak hyperbolicity. The latter was associated with essentially ensuring that $c_s^2 > 0$, and that the system did not admit gradient instabilities. The former, on the other hand, was only satisfied if the system had a complete set of eigenvectors, which were finite and depended smoothly on the initial conditions. This would imply that the solution would have an upper energy bound, and that

[3] For high energy modes, this bound gets modified as irrelevant operators dominate the dispersion relation; see Section 3 of [108] for a detailed explanation.

the modes would be "under perturbative control". Moreover, it was claimed that satisfying the strong hyperbolicity condition would ensure "robustness against arbitrarily small wavelength perturbations". If we attempt to understand this in the context of [108], this would mean that an arbitrarily small $B(t)$ (as is shown to occur in Figure 3 of [33]) would not present a major problem. However, we must take care with this argument, since it remains a largely classical one, and it assumes that NEC violation definitely occurred. The potential problem that remains is that, if unitarity were violated (at least at tree-level), the validity of the aforementioned assumption would be brought into question, depending on the relation between the background and strong coupling scales. Said differently, an arbitrarily small $B(t)$ implies that one should re-assess the validity of the conclusions reached by [33]. This would be achieved by considering the quantised description of the perturbed scalar sector and finding the partial wave expansion of the scattering amplitudes. This would have to be weighed against the requirements set by perturbative unitarity, which would then allow us to reach a conclusion as to whether the latter does actually present a problem or not. We contrast the expected resolution of this problem against that of the instability of the background trajectory, which was discussed in Section 4.1. For that issue, it was found that, in the case of two decoupled fields, having $s \neq 0$ (even infinitesimally) meant that the background would rapidly evolve away from the stable fixed point which would allow a stable cyclic evolution. That could be resolved by including an exponential coupling and showing that this led to the existence of a stable fixed-curve solution, to which the trajectory would be attracted instead. The genesis of the problem discussed here is entirely different, and so we cannot expect its resolution to be similar, in terms of a statement about the hyperbolicity or stability of the PDE system.

We note that some steps have recently been taken in this regard. Specifically, the question of perturbative unitarity in a Horndeski/beyond-Horndeski setup was investigated in [110], albeit with the cosmological model in question being Galilean genesis (another alternative to inflation, involving an NEC-violating period which causes expansion from Minkowski spacetime in the far past [111]). In this case, the Horndeski theory leading to the Galilean genesis solution was taken as an EFT, with the higher-order operators being of beyond-Horndeski form. Scattering amplitudes were explicitly calculated in the far past limit, and it was found that perturbative unitarity was violated at an energy scale which was much less than the cut-off scale of the EFT action, $\sqrt{s} \ll M$. As a result, the model used by the authors would require other operators to restore unitarity well inside the realm of the EFT.

As attractive as adapting this calculation straightaway to the context of the new cyclic model might sound, there were two issues that we found to considerably increase the difficulty of the computation. Firstly, the analysis of [110] was conducted in spatially flat gauge, which was sufficiently suitable for the purpose of the genesis model and relatively simple to work with (since it allows one to focus directly on the perturbations in the scalar field). However, as mentioned earlier and shown in [32], this gauge introduces unwanted behaviour as a coordinate singularity manifests itself; $\gamma(t) \to 0$, which leads to the gauge variables α and σ going to infinity. The Newtonian gauge $\sigma \equiv 0$ is the unique algebraic gauge that removes this problem, but focusing on the behaviour of the scalar perturbation φ becomes less straightforward[4]. Secondly, the couplings used in the genesis model were constants, which meant that they appeared in the perturbed Lagrangian just as in the full one. On the other hand, we recall that the cyclic model presented here involved couplings that were functions of the scalar, and their non-trivial behaviour (especially in the case of $\mathcal{L}_4$) was essential to maintaining $c_s^2 > 0$. This means that, to perturb the action, one would have to consider the Taylor expansions of these coupling functions to the desired order, in order to find the Feynman diagrams and compute the scattering amplitudes. While we have outlined the method we would expect to use, the joint impact of these complications means that a full calculation of the latter, and a complete analysis of the impact of perturbative unitarity at tree level on the bounce phase of the new cyclic model is outside the scope of this dissertation.

However, in light of both [108] and [110], we can still attempt to provide some direction in the sense of what we would expect the result to be. Looking at the form of the bounds derived in previous analyses, the right hand side is in terms of the coupling scale that multiplies the higher-order operator(s). Hence, we would expect unitarity to place restrictions on the coupling functions (and possibly their derivatives), thus constraining their behaviour more stringently than implied by the forms of the input parameters and the relations (4.2.20)–(4.2.23) resulting from the background equations (4.2.6)–(4.2.7). As though confirming our line of thought, we see that a similar analysis has been carried out in [112], where the authors consider the impact of perturbative unitarity violation in the early contracting phase of a Horndeski bouncing model with strong gravity in the distant past, when scalar perturbations are generated. While the cosmological epoch under consideration in that work is different, it is found that the requiring the strong coupling scale to be greater than the background scale results in the tensor-to-scalar ratio r being bounded from below. Therefore, fine-tuning in the new cyclic model of Ijjas and Steinhardt is not quite non-existent in the sense that [31] would seem to imply; whether it conceptually presents as large a problem as in inflation is a question that can only be answered with a thorough investigation of this issue.

[4] A possible workaround might be to focus on Feynman diagrams generated with $\mathcal{R}$ and h in the unitary gauge and calculate the resulting scattering amplitudes, as in [112].

4.5 Discussion

In this chapter, we presented the new form of the cyclic model of the Universe proposed by Ijjas, Steinhardt *et. al.*, which has come about in recent years. The first significant improvement with respect to the previous models was made by introducing an exponential coupling to the canonical kinetic term of the second scalar field, so that the background instability problem shown to exist in a contracting Universe by Tolley and Wesley was resolved. The introduction of a quartic kinetic term (also featuring an exponential coupling) was shown to facilitate the conversion of a nearly scale-invariant spectrum of entropic perturbations into that of curvature perturbations in the kinetic-dominated phase following ekpyrotic contraction. The second advance was made by finding a mechanism that admitted a non-singular bounce which appears to be stable and pathology-free for all finite times. This was constructed in the framework of a generalised Galileon/Horndeski theory using derivative operators up to and including quartic order, and employing an 'inverse method' to show that ghosts and gradient instabilities were avoided. We also discussed the evolution of entropy in this new model and showed that it represents a significant advantage over the corresponding problem posed in inflation.

On the other hand, we have also identified some outstanding issues. First, the bounce is expected to take place at sub-Planckian scales, with the Hubble parameter rapidly increasing from a large negative to a large positive value. However, there is nothing in the framework of the theory to suggest that there is a hard cut-off, preventing $H(t)$ from oscillating between large and finite, but Planck-scale values. Indeed, the assumption of a sub-Planckian maximum $|H(t)|$ is inserted into the model by construction and it would be interesting to find a physical explanation that allows this to occur naturally. We note that Ijjas and Steinhardt in [31] conjectured a generalised cosmic censorship principle, where the evolution of the Universe would be shielded from the domination of quantum over classical effects. A theoretic explanation for the aforementioned requirement would likely arise from the same source as this conjecture, if it were to hold true.

Second, the discussion of entropy evolution involved a major role being played by black holes containing the dominant contribution to the increase in entropy over the course of the expanding phase. While non-perturbative studies have been carried out to understand the behaviour of black holes in the near-field regime of a Hubble patch undergoing a non-singular bounce, one question that remains open is what happens when the minimum size of the Hubble radius is much smaller than the radius of the black hole, in the specific context of the Horndeski theory.

Third, but certainly not least, we considered the impact of requiring perturbative unitarity to hold during the NEC-violating phase. The motivation for this was derived from [108], which investigated this in the context of $P(X)$ theories. While a related problem was recognised in [100], its resolution did not map directly to a resolution of the unitarity violation problem. Given recent related work in the context of a Galilean genesis scenario and of the early contracting phase in an independent Horndeski-based bouncing model, we established expectations of the consequences that imposing unitarity at tree level would have on the theory. We hope to be able to delve further into this problem, and present results of our investigation in the future.

Conclusion

5

The science of cosmology has come a long way in the last fifty years. While General Relativity opened our eyes to (a more complete picture of) the large-scale dynamics of gravitation, the hot Big Bang model that resulted from applying it to the evolution of the Universe was known to suffer from the horizon, flatness, large-scale structure formation and origin of the Universe problems. In this dissertation, we have demonstrated the various means by which resolutions to these problems have been proposed.

After introducing the problems known to face hot Big Bang cosmology, we presented the inflationary paradigm in Chapter 2 as the first method that sought to answer these consistently. We looked at the dynamics of a scalar field in a slow-roll potential and derived the associated slow-roll conditions. Working in the unitary gauge, we then found the form of the power spectrum of nearly scale-invariant curvature perturbations by deriving the Mukhanov-Sasaki equation and solving for the quantised mode fluctuations. This was done to present the resolution to the large-scale structure formation problem from the point of view of inflation. Perturbations in the tensor sector were shown to lead to predictions of primordial gravitational waves, with the consistency relation $r = -8n_t$ providing a vital link between the observable tensor-to-scalar amplitude ratio and the tensor spectral index. We then considered a specific example of embedding inflation within the fundamental setup of M-theory, which was boundary inflation. Since the required ingredients are simply assumed to exist in the basic scenario, this was done so that we could understand how inflation could possibly arise in the primordial Universe. We concluded our consideration of inflation by understanding some of the problems that were known to exist before and those that have come to light in recent years.

Chapter 3 introduced the possibility of bouncing cosmologies as an alternative to cosmic inflation. Specifically, we focused on the cyclic/phoenix model of Turok, Steinhardt *et. al.* which used the same fundamental M-theory construction as the boundary inflation case, albeit to yield an entirely different effect. We presented the background dynamics of a scalar field in an ekpyrotic contracting phase which transitioned to an expanding phase via a singular bounce. However, this singularity was shown to be milder than in the case of the consensus Λ-CDM plus inflation picture, since it only involved the momentary disappearance and re-appearance of the fifth orbifold dimension, along which the boundary planes representing the visible Universe and 'hidden' sector were moving and colliding. This was generalised to multiple scalars, which was particularly significant for the reason that the isocurvature/entropic mechanism describing the formation of a nearly scale-invariant spectrum of curvature perturbations required two scalars to match observations. Moreover, dark energy was shown to play a critical role in this model (in contrast to Λ-CDM cosmology plus inflation) as the key that triggered the smooth transition from expansion to contraction. Also, given the background instability that the isocurvature mechanism brought about, a sufficiently long period of dark energy domination ensured that a sufficiently large fraction of the visible Universe would continue the cyclic trajectory, reincarnating from the ashes like a phoenix. We then showed how tensor-sector perturbations in this model yielded the first departure from the predictions of inflation, by implying a spectrum of gravitational waves that would be negligible in the present day. We briefly looked at alternative contructions of bouncing cosmologies, specifically the matter bounce and pre-Big Bang scenarios, and closed by understanding the problems facing these models.

In Chapter 4, we presented a radically different approach to a cyclic Universe, although we dedicated the opening section to discussing a resolution of the background instability problem that was not necessarily unique to the new model. Specifically, we showed how the introduction of an exponential coupling in the two-scalar Lagrangian in the contracting phase implied the existence of a stable fixed-curve solution, to which the background would be attracted for generic initial conditions. We then considered the robustness of ekpyrotic contraction after the modification of the kinetic term, and found that a broken shift symmetry would be required, in the form of a quadratic potential and a small non-zero mass for the second scalar. The use of a quartic kinetic term was then shown to be relevant as facilitating the conversion of entropic to curvature perturbations in the post-contraction kinetic-dominated phase. We proposed that this might also be the physical motivation for the potential asymptoting back to zero after the end of contraction. At this junction, we shifted our perspective to understanding the viability of a non-singular bounce that could be described in classical terms. After showing that this necessitated the use of (amongst other possibilities, such as modified matter) a modified gravity theory, we analysed the specific example shown by Ijjas and Steinhardt to avoid ghosts and gradient instabilities for all finite times. This was achieved using a generalised Galileon/Horndeski framework, involving nontrivial contributions up to and including quartic order in the derivatives of the scalar field. After a discussion of the evolution of entropy in the context of this new cyclic Universe, we turned our attention to the question of perturbative unitarity and its possible violation during the NEC-violating bounce. We drew inspiration from the previous analysis of de Rham and Melville in the context of $P(X)$ theories, and proposed that requiring unitarity to hold would constrain the coupling functions more than implied in the existing literature. In other words, to ensure the consistency of the description of the bounce at the quantum level, an additional dose (possibly) of fine-tuning would be required.

While we are excited by the possibilities brought about in this new era of cosmology, we realise that there is a long way to go before the theoretical issues underlying the models investigated in this dissertation can be fully resolved. In this regard, however, we find great opportunity for future advances to take place and hope to partake in some of them.

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